Robust Allocation of Testing Resources in Reliability Growth

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Abstract

Reliability growth testing seeks to improve system reliability by identifying and removing failure modes. Recent models maximize system reliability by allocating limited testing resources across the system’s components, each of which exhibits reliability growth according to the AMSAA model (Crow, 1974) with known parameters. We extend this research to solve a robust version of this problem for both series and series-parallel systems in which AMSAA parameters are uncertain but assumed to lie within a budget-restricted uncertainty set. We develop and analyze an exact solution approach for this problem based on a cutting-plane algorithm. In the case of series systems, we demonstrate that the subproblems from this algorithm are efficiently solvable via dynamic programming. Computational results demonstrate (i) the value of the robust optimization approach as compared to deterministic alternatives and (ii) the efficacy of our algorithm.

1 Introduction

The improvement in system reliability by identifying and removing the failure modes of design is known as reliability growth. Reliability growth occurs when operating conditions are simulated during developmental tests to identify and remove failure modes. Testing each component in a system results in different costs and/or benefits to the system, thus motivating the problem of how to allocate limited testing resources among components. Previous research addresses the case where the relationship between component testing and component reliability is known (deterministically), but there is a lack of work addressing what happens when this relationship is subject to uncertainty. This is problematic because reliability growth models are built upon preliminary failure data that is subject to randomness. In this paper, we consider a robust allocation of testing resources across the components of the series/series-parallel system to increase system reliability in this environment.

Reliability growth has received significant attention over the past 50 years. Duane [14] proposed a reliability growth model, which is based on the observation that the logarithm of cumulative

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mean time between failures (MTBF) has a linear relationship to the logarithm of cumulative test time. The reliability growth concept was further studied by Crow [11], who proposed the Army Material Systems Analysis Activity (AMSAA) model that remains among the most commonly-used reliability growth models. Crow showed if the Duane model holds, then failures can be represented as a nonhomogeneous Poisson process with Weibull intensity. Thus, in the AMSAA model, improvement in the reliability is modeled via a decreasing intensity function. In this model, there are no specific assumptions about the number and type of failure modes, therefore making the model useful for a variety of systems.

A number of reliability growth models have been developed over the years. We review some of significant contributions here but refer the interested reader to [19, 38] for a comprehensive survey of significant works. Discrete reliability growth models (e.g., [21, 31]) use the countable data such as number of failures or pass/fail tests, while continuous reliability growth models (e.g., [11, 12, 14, 20, 30]) use continuous data, typically time. Reliability growth models consider both hardware (see, e.g., [11, 12, 14, 31]) and software [20, 37] systems, and they have incorporated accelerated testing [26], Bayesian (e.g., [27, 29]) and nonparametric [34, 35] methods.

In this research, we consider the robust allocation of testing resources in reliability growth under the assumption that components exhibit reliability growth according to the AMSAA model. Related optimization models date back to the work of Coit [10], who considered the allocation of testing resources for a series system under the assumption that component failures occur according to the AMSAA model. These assumptions yield a convex optimization problem, which can be solved to optimally using standard techniques. Heydari et al. [23], Dai et al. [13] and Levitin [28] re-examine this problem in the case of a series-parallel system, and Heydari et al. [24] demonstrate that this class of models can be extended to consider subsystem- and system-level testing. Heydari et al. [22] later extend this problem to incorporate redundancy allocation decisions. In the case of the series-parallel systems with fixed redundancy levels, the convex optimization property no longer holds, leading to computationally difficult models that have been solved exactly [22, 23] and heuristically [13, 28, 24].

With a few exceptions that we now summarize, all of these resource allocation papers assume known AMSAA parameters, thus yielding deterministic optimization models. However, reliability growth testing is typically applied to new systems with unproven designs [10], and for these systems, there is insufficient data to estimate the AMSAA model parameters with high accuracy, thus introducing uncertainty into the resource allocation problem. Awad [1] and Coit [10] address this limitation to allocate testing times in a series system when each component’s failure rate is normally distributed having a mean given by an AMSAA model with known parameters. By contrast, our robust optimization approach needs not assume any underlying probability distribution. Moreover, our method can be extended to consider the allocation of testing times, for the first time in the literature, in a series-parallel system when component reliability growth has AMSAA model
structure with unknown parameters.

Robust optimization seeks to generate a solution in which its objective value or its feasibility must be guaranteed for any realization of its parameters within a bounded uncertainty set. Robust linear optimization was first examined by Soyster [36] and later extended (by Ben-Tal and Nemirovski [3] and El-Ghaoui et al. [15]) to derive efficient algorithms for robust convex optimization problem. Bertsimas and Sim [7] characterize a class of robust discrete optimization problems that remain efficiently solvable. Robust optimization has variety of application in decision making environments such as linear programming, assignment problem, shortest paths, minimum spanning tree, knapsack problem, resource allocation, scheduling, production planning, inventory, layout planning and network design [25]. We refer the interested reader to Ben-Tal and Nemirovski [4] and Bertsimas et al. [6] for comprehensive surveys of robust optimization theory and applications.

This paper is organized as follows: the background and problem definition is defined in Section 2. We present a robust reliability growth model for a series system in Section 3 and for a series-parallel system in Section 4. Computational results are summarized in Section 5, and we conclude in Section 6.

2 Background and Problem Definition

Following models in [10, 13, 23, 28], we consider the problem of allocating limited testing times to independent components within a system. Let \( N \) denote the number of subsystems in the system, and let the \( N \)-vector of decision variables \( \tau \) represent the total testing time (in hours) allocated to each subsystem. One hour of testing on subsystem \( \ell = 1, \ldots, N \) is assumed to require \( c_{v}^{\ell} \) units of resource \( v = 1, \ldots, V \). The \( N \)-vector \( \tau^{0} \) denotes the preliminary testing time, where the preliminary testing time \( \tau^{0}_{\ell} \) can be interpreted as the amount of testing already completed on subsystem \( \ell \) at the time our optimization model is solved to generate a test plan for the system. This preliminary testing may be used to estimate AMSAA model parameters (and their associated uncertainty) that are used to build the optimization model.

The objective in allocating limited testing time is to maximize system reliability over a mission of length \( t > 0 \) hours that begins after testing. The deterministic allocation problem is generally stated as

\[
\begin{align*}
\max & \quad R(\tau, t), \\
\text{s.t.} & \quad \tau \in X \equiv \left\{ \tau \left| \sum_{\ell=1}^{N} c_{v}^{\ell} \tau_{\ell} \leq b_{v}, \quad v = 1, \ldots, V; \quad \tau \geq \tau^{0} \right. \right\}
\end{align*}
\]

where \( R(\tau, t) \), which is defined mathematically in the following paragraph, denotes the system’s mission reliability resulting when subsystems are tested according to \( \tau \). Constraints (1b) ensure
that (i) testing time for each subsystem \( \ell \) is at least its initial testing time \( \tau_\ell^0 \) and (ii) the total resource consumption across all subsystems is no more than \( b_v \) for all resources \( v = 1, \ldots, V \). We initially assume (for mathematical convenience) that \( \tau_\ell^0 \geq 1 \), \( \forall \ell = 1, \ldots, N \) but will address (at the end of Section 3) recourse in the event this assumption does not hold. In what follows, we extend Model (1) to consider uncertainty in \( R(\tau, t) \) via robust optimization. Before doing so, we first summarize assumptions and existing results for Model (1) that are pertinent to our research.

As in \[10, 13, 22, 23, 28\], we relate system reliability to subsystem test times \( \tau \) by assuming that subsystems are collectively independent of other and each exhibits reliability growth according to the AMSAA model. Under this model, failures of a component in subsystem \( \ell = 1, \ldots, N \) occur during a testing period of length \( \tau_\ell > 0 \) according to a nonhomogenous Poisson process with decreasing rate given by

\[
u(\tau_\ell; \lambda_\ell, \beta_\ell) = \lambda_\ell \beta_\ell \tau_\ell^{\beta_\ell - 1},
\]

where \( \lambda_\ell > 0 \) and \( 0 < \beta_\ell < 1 \). After \( \tau_\ell \) hours of testing have been completed on subsystem \( \ell \), failures of a component in subsystem \( \ell = 1, \ldots, N \) occur according to a homogenous Poisson process with constant rate \( \nu(\tau_\ell; \lambda_\ell, \beta_\ell) \), so that the component reliability for a mission of length \( t > 0 \) hours is given by

\[
R(\tau_\ell; t, \lambda_\ell, \beta_\ell) = e^{-\nu(\tau_\ell; \lambda_\ell, \beta_\ell)t} = e^{-\lambda_\ell \beta_\ell \tau_\ell^{\beta_\ell - 1}t}.
\]

The objective function \( R(\tau, t) \) from Model (1) is then related to the growth functions \( R(\tau_\ell; t, \lambda_\ell, \beta_\ell) \) according to the system’s structure, and we hereafter denote this objective as \( R(\tau; \lambda, \beta) \) to note its dependence on parameter vectors \( \lambda \) and \( \beta \). (Boldface \( \lambda \) and \( \beta \) are used to represent the associated \( N \)-vectors of \( \lambda_\ell \)- and \( \beta_\ell \)-values.) Thus, if the \( N \) one-component subsystems are connected in series, then

\[
R(\tau; \lambda, \beta) = \prod_{\ell=1}^{N} R(\tau_\ell; t, \lambda_\ell, \beta_\ell),
\]

but if each subsystem \( \ell = 1, \ldots, N \), has \( M_\ell \) identical components that are simultaneously improved by increasing \( \tau_\ell \), then

\[
R(\tau; \lambda, \beta) = \prod_{\ell=1}^{N} \left[ 1 - (1 - R(\tau_\ell; t, \lambda_\ell, \beta_\ell))^{M_\ell} \right].
\]

Given the definitions above, it is now appropriate to describe the robust optimization problem considered in this paper. Our problem differs from Model (1) because we do not assume \( \lambda \) and \( \beta \) are known with certainty. Rather, we assume \((\lambda, \beta)\) lies within an uncertainty set, \( U \), and consider the (robust) optimization problem

\[
\max_{\tau \in X} \min_{(\lambda, \beta) \in U} R(\tau; \lambda, \beta).
\]

Throughout the reminder of this document, we refer to a pair \((\lambda, \beta) \in U\) as a scenario, and we
refer to \( \min_{(\lambda, \beta) \in U} R(\tau; \lambda, \beta) \), the reliability at \( \tau \in X \) that corresponds to the worst-case scenario, as the robust objective value. Note that Model (6) maximizes the robust objective value over all feasible test strategies.

The models throughout the remainder of this paper utilize a so-called budgeted uncertainty set. (The interested reader may refer to [7, 8, 9, 16, 18] for other examples of budgeted uncertainty and [16, 17, 18] for reliability-related robust optimization models.) We assume each parameter is restricted to fall within an interval, i.e.,

\[
\lambda_\ell \in [\bar{\lambda}_\ell, \bar{\lambda}_\ell + \sigma^L_\ell], \quad \ell = 1, \ldots, N, \tag{7a}
\]

\[
\beta_\ell \in [\bar{\beta}_\ell, \bar{\beta}_\ell + \sigma^B_\ell], \quad \ell = 1, \ldots, N. \tag{7b}
\]

We scale each interval by introducing new parameters \( \gamma^B_\ell \) and \( \gamma^L_\ell \) such that \( \beta_\ell = \bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell \) and \( \lambda_\ell = \bar{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell \), in which \( \bar{\lambda}_\ell \), \( \sigma^L_\ell \geq 0 \) and \( \bar{\beta}_\ell \), \( \bar{\beta}_\ell + \sigma^B_\ell \) are in \([0, 1] \). Using the scaled parameters, we define the uncertainty set \( U_\Phi \) for a given budget \( 0 \leq \Phi \leq 2N \) as

\[
U_\Phi = \left\{ (\gamma^L, \gamma^B) \in \mathbb{R}^N : \sum_{\ell=1}^N (\gamma^L_\ell + \gamma^B_\ell) \leq \Phi, 0 \leq \gamma^L_\ell, \gamma^B_\ell \leq 1 \right\}. \tag{8}
\]

As we will show in Theorem 1, the reliability function is decreasing in \( \gamma^L_\ell \) and \( \gamma^B_\ell \); therefore, \( \Phi = 0 \) represents an optimistic view of uncertainty (i.e., all parameters will take their best-case values, which happen to be their smallest values) and \( \Phi = 2N \) is pessimistic in that all parameters will take their worst-case/largest values. We restate Model (6) as

\[
\max_{\tau \in X} \min_{(\gamma^L, \gamma^B) \in U_\Phi} R \left( \tau; \lambda + \sigma^L \gamma^L, \beta + \sigma^B \gamma^B \right), \tag{9}
\]

where \( \ast \) denotes element-wise multiplication of the associated \( N \)-vectors. Our reasoning for employing a single uncertainty parameter \( \Phi \) (e.g., instead of having two parameters to enable separate uncertainty budgets for \( \lambda \) and \( \beta \)) is two-fold: (i) each uncertainty parameter adds an additional dimension to the set of policies that can be generated from our model, thereby increasing the complexity associated with determining how to best implement the model; and (ii) our integrality property (Theorem 2, proven in the following section), which enables efficient solution of the robust optimization problem for series systems, fails in the case of a multi-parameter uncertainty set.

To summarize, our major assumptions are that the system structure is fixed, subsystems are independent, redundant components are in active standby, and the mission length \( t \) is fixed. Moreover, it is assumed that the failures of subsystems occur according to the AMSAA reliability growth model, in which parameters \( \lambda_\ell \) and \( \beta_\ell \) lie within a budget-restricted uncertainty set. Although we believe extensions of this paper may lead to weakening some of these, all of these assumptions are limiting for the results given in this paper. In addition to the assumptions listed above, we have
also assumed that $\tau_{\ell} \geq 1$, $\ell = 1, \ldots, N$. This assumption is purely for mathematical convenience in the case of series system, and we have discussed (in the last paragraph of Section 3) a means of solving the problem in the event this assumption does not hold.

Before developing the mechanics to solve Model (6), we first summarize its relation to important results from the literature. Coit [10] considered a version of Model (1) in which components are connected in a series, which leads to a convex optimization problem. As we see in Section 3, the robust version of this problem turns out to be easily solvable as well. On the other hand, when the system is series-parallel, Heydari et al. [23] propose exact algorithms for the deterministic model. We extend the deterministic series-parallel model into a robust model (see Section 4) and demonstrate that the resulting problem can be solved via extension of the algorithm given in [23].

The notation used in the remainder of the document is summarized below:

**Parameters**

\[
\begin{align*}
N & \quad \text{Number of subsystems} \\
t & \quad \text{Mission length} \\
V & \quad \text{Number of resources} \\
b_v & \quad \text{Amount of resource } v = 1, \ldots, V \text{ available} \\
c_v^\ell & \quad \text{Amount of resource } v = 1, \ldots, V \text{ required per unit time of testing subsystem } \ell = 1, \ldots, N \\
\beta_{\ell}, \lambda_{\ell} & \quad \text{AMSAA model parameters for subsystem } \ell = 1, \ldots, N \\
[\bar{\beta}_{\ell}, \bar{\beta}_{\ell} + \sigma_{\ell}^B] & \quad \text{Interval for parameter } \beta \text{ of subsystem } \ell = 1, \ldots, N \\
[\bar{\lambda}_{\ell}, \bar{\lambda}_{\ell} + \sigma_{\ell}^L] & \quad \text{Interval for parameter } \lambda \text{ of subsystem } \ell = 1, \ldots, N \\
\gamma_{\ell}^L, \gamma_{\ell}^B & \quad \text{Realized coefficient for the AMSAA model parameters for subsystem } \ell = 1, \ldots, N \\
\Phi & \quad \text{Uncertainty budget } (\Phi \in [0, 2N]) \\
U_\Phi & \quad \text{Uncertainty set} \\
\tau_{\ell}^0 & \quad \text{Initial testing time of subsystem } \ell = 1, \ldots, N \text{ (assumed to be greater or equal to 1)} \\
P_{\ell} & \quad \text{Number of discrete points for subsystem } \ell = 1, \ldots, N \\
P & \quad \text{Number of discrete points for each subsystem (assumed to be equal for all subsystems)} \\
r_{\ell}^p & \quad \text{Reliability of point } p \text{ in subsystem } \ell \text{ (} \ell = 1, \ldots, N; p = 1, \ldots, P_{\ell} \text{)} \\
r_{\ell}^p & \quad \text{The value of } \gamma_{\ell}^L + \gamma_{\ell}^B \text{ for point } p \text{ in subsystem } \ell \text{ (} \ell = 1, \ldots, N; p = 1, \ldots, P_{\ell} \text{)} \\
r_{\ell}^p & \quad \text{Testing time for point } p \text{ in subsystem } \ell \text{ (} p = 1, \ldots, P; \ell = 1, \ldots, N \text{)} \\
r_{\ell,j}^p & \quad \text{Reliability of point } p \text{ in subsystem } \ell \text{ for scenario } j \text{ (} p = 1, \ldots, P; \ell = 1, \ldots, N; j = 1, \ldots, |\bar{D}| \text{)}
\end{align*}
\]
Decision Variables

\[ \tau_\ell \quad \text{Testing time of subsystem } \ell = 1, \ldots, N \]

\[ x^p_\ell \begin{cases} 
  1 & \text{if the value of } \gamma^L_\ell + \gamma^B_\ell \text{ in subsystem } \ell \text{ is } z^p_\ell \ (\ell = 1, \ldots, N; \ p = 1, \ldots, P) \\
  0 & \text{otherwise}
\end{cases} \]

\[ y^p_\ell \begin{cases} 
  1 & \text{if the testing time for subsystem } \ell \text{ is } \tau^p_\ell \ (p = 1, \ldots, P; \ \ell = 1, \ldots, N) \\
  0 & \text{otherwise}
\end{cases} \]

In the following section, we specialize Model (6) for the case of series system and develop a cutting-plane algorithm to solve it.

### 3 Solution Method for Series Systems

We now specialize the problem to the case of series structure. Let \( N \) define the number of subsystems (each with a single component) in the series system. Based on Equation (4), \( R(\tau; \lambda, \beta) \) is given by

\[
R(\tau; \lambda, \beta) = \prod_{\ell=1}^{N} R(\tau^\ell; t, \lambda^\ell + \sigma^L_\ell \gamma^L_\ell, \beta^\ell + \sigma^B_\ell \gamma^B_\ell), \quad (10)
\]

in which the uncertainty set \( U_\Phi \) is defined in Equation (8). For series systems, the robust optimization Model (9) can be stated as

\[
\begin{align*}
\max & \quad R, \\
\text{s.t.} & \quad \prod_{\ell=1}^{N} e^{-(\lambda^\ell + \sigma^L_\ell \gamma^L_\ell)(\beta^\ell + \sigma^B_\ell \gamma^B_\ell) t} \geq R, \quad \forall (\gamma^L, \gamma^B) \in U_\Phi, \quad (11b) \\
& \quad \tau \in X. \quad (11c)
\end{align*}
\]

Towards solving Model (11), we now propose a cutting-plane algorithm to solve Model (11) optimally. (The interested reader may refer to [5] for seminal work on cutting-plane algorithms for min-max and max-min problems.) Model (11) is a nonlinear/nonconvex model, and Constraints (11b) are uncountably infinite, so Model (11) is difficult to solve directly. By applying a logarithm to Constraint (11b), Model (11) is equivalent to

\[
\begin{align*}
\max & \quad R, \\
\text{s.t.} & \quad \sum_{\ell=1}^{N} -(\lambda^\ell + \sigma^L_\ell \gamma^L_\ell)(\beta^\ell + \sigma^B_\ell \gamma^B_\ell) t \geq \ln R, \quad \forall (\gamma^L, \gamma^B) \in U_\Phi, \\
& \quad \tau \in X. \quad (12c)
\end{align*}
\]
Model (12) turns out to be convex, but we are still left with the issue that (12b) is indexed over an infinite set of constraints. We hence apply a cutting-plane algorithm to solve Model (12). The separation model, used to compute the robust objective value for given \( \tau \in X \), can be stated as

\[
G(\tau) = \min \left\{ F(\gamma^L, \gamma^B) \mid (\gamma^L, \gamma^B) \in U_\Phi \right\},
\]

(13)

where \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \equiv -(\tilde{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\tilde{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell) \tau^\ell \) and \( F(\gamma^L, \gamma^B) \equiv \sum_{\ell=1}^N f_\ell(\gamma^L_\ell, \gamma^B_\ell) \). The solution for Model (13) generates a feasible solution for Model (11) with \( R = e^{G(\tau)} \); therefore, Model (13) provides a lower bound for the logarithm of the objective value of Model (11) (i.e., \( e^{G(\tau)} \leq R^* \)). Model (13) is nonconvex in general, but we now demonstrate that it can be solved efficiently via a dynamic programming algorithm. We begin by establishing that \( F(\gamma^L, \gamma^B) \) is monotone and componentwise concave, i.e., concave in either \( \gamma^L_\ell \) or \( \gamma^B_\ell \) whenever all other \( \gamma^L \)- and \( \gamma^B \)-variables are fixed.

**Theorem 1.** The function \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) is decreasing and componentwise concave.

**Proof.** The partial derivatives of \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) are given by

\[
\frac{\partial f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial \gamma^L_\ell} = -\sigma^L_\ell \left( \tilde{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell \right) \tau^\ell \gamma^B_\ell - 1 \tau^\ell,
\]

(14a)

\[
\frac{\partial f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial \gamma^B_\ell} = -\sigma^B_\ell \left( \tilde{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell \right) \tau^\ell \ln(\tau^\ell).
\]

(14b)

Therefore, \( \frac{\partial f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial \gamma^L_\ell} < 0 \) and the assumption \( \tau^\ell > 1 \) implies \( \frac{\partial f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial \gamma^B_\ell} < 0 \) in Equation (14b) as well. As a result, \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) is decreasing regarding \( \gamma^L_\ell \) and \( \gamma^B_\ell \) when the other is fixed. The second derivatives of \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) with respect to variables \( \gamma^L_\ell \) and \( \gamma^B_\ell \) are equal to

\[
\frac{\partial^2 f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial^2 \gamma^L_\ell} = 0,
\]

(15a)

\[
\frac{\partial^2 f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial^2 \gamma^B_\ell} = -(\sigma^B_\ell)^2 \left( \tilde{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell \right) \ln(\tau^\ell) \left[ 2 + \left( \tilde{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell \right) \ln(\tau^\ell) \right] \tau^\ell.
\]

(15b)

Note that \( \frac{\partial^2 f_\ell(\gamma^L_\ell, \gamma^B_\ell)}{\partial^2 \gamma^B_\ell} \leq 0 \), since \( \ln \tau^\ell \geq 0 \). Therefore, \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) is also componentwise concave.

\[\Box\]

**Remark 1.** Because \( F(\gamma^L, \gamma^B) \) is a summation of terms that all utilize a different \( \ell \)-index, Theorem 1 also proves that the function \( F(\gamma^L, \gamma^B) \) is decreasing and componentwise concave.

As we demonstrate in Theorems 4–5, the previous result allows us to guarantee existence of an optimal solution to Model (13) among a finite set of candidate solutions. In the following counterexample, we demonstrate that \( f_\ell(\gamma^L_\ell, \gamma^B_\ell) \) is not concave in general.
Counterexample 1. Define $\hat{\lambda}_\ell = 0.00001$, $\sigma^L_\ell = 0.00006\beta_\ell = 0.6$, $\sigma^B_\ell = 0.15$, $\tau_\ell = 100$, $V = 1$, $c^1_\ell = 1$ and $b_1 = 8760$. Letting $(\hat{\gamma}^L_\ell, \hat{\gamma}^B_\ell) = (1, 0)$, $(\hat{\tilde{\gamma}}^L_\ell, \hat{\tilde{\gamma}}^B_\ell) = (0, 1)$, and $\alpha = 0.5$, we have

\[
\begin{align*}
    f_\ell(\alpha\hat{\gamma}^L_\ell + (1 - \alpha)\hat{\gamma}^B_\ell, \alpha\hat{\gamma}^L_\ell + (1 - \alpha)\hat{\gamma}^B_\ell) &= -0.05295, \\
    \alpha f_\ell(\hat{\gamma}^L_\ell, \alpha\hat{\gamma}^B_\ell) + (1 - \alpha)f_\ell(\hat{\tilde{\gamma}}^L_\ell, \hat{\tilde{\gamma}}^B_\ell) &= -0.03954.
\end{align*}
\]

Because $f_\ell(\alpha\hat{\gamma}^L_\ell + (1 - \alpha)\hat{\gamma}^B_\ell, \alpha\hat{\gamma}^L_\ell + (1 - \alpha)\hat{\gamma}^B_\ell) < \alpha f_\ell(\hat{\gamma}^L_\ell, \alpha\hat{\gamma}^B_\ell) + (1 - \alpha)f_\ell(\hat{\tilde{\gamma}}^L_\ell, \hat{\tilde{\gamma}}^B_\ell)$, the function $f_\ell(\gamma^L_\ell, \gamma^B_\ell)$ is not concave in general.

Theorem 2. There exists an optimal solution to Model (13) such that the number of subsystems $\ell$ with $\gamma^L_\ell$ and/or $\gamma^B_\ell$ taking on noninteger values is at most 1.

Proof. Let $(\hat{\gamma}^L, \hat{\gamma}^B)$ denote an optimal solution to Model (13). Among all such solutions, let $(\hat{\gamma}^L, \hat{\gamma}^B)$ be an optimal solution with the minimum number of subsystems $\ell$ such that $\gamma^L_\ell$ and/or $\gamma^B_\ell$ is fractional. (In the remainder of this proof, we refer to such a subsystem as a fractional subsystem.) Let $k^*$ denote the number of fractional subsystems in $(\hat{\gamma}^L, \hat{\gamma}^B)$. If $k^* = 0$ or 1, there is nothing to prove, so suppose $k^* \geq 2$. We establish a contradiction by constructing an optimal solution to Model (13) with only $k^* - 1$ fractional subsystems. First, let $\{\ell', \ell''\}$ denote a pair of distinct fractional subsystems with respect to $(\hat{\gamma}^L, \hat{\gamma}^B)$ in which at least one of $\{\hat{\gamma}^L_\ell, \hat{\gamma}^B_\ell\}$ is noninteger for $\ell \in \{\ell', \ell''\}$. At least one of the following three cases will hold.

Case 1: Suppose $\hat{\gamma}^L_{\ell'}$ and $\hat{\gamma}^B_{\ell''}$ are fractional. Define $z = \hat{\gamma}^L_{\ell'} + \hat{\gamma}^B_{\ell''}$. When $\gamma^L_\ell = \hat{\gamma}^L_{\ell'}$, $\ell \in \{1, \ldots, N\} \setminus \{\ell', \ell''\}$ and $\gamma^B = \hat{\gamma}^B$ are fixed in Model (13), it reduces to the linear program

\[
\begin{align*}
    \min & \quad f_{\ell'}(\gamma^L_{\ell'}, \hat{\gamma}^B_{\ell''}) + f_{\ell''}(\gamma^L_{\ell'}, \hat{\gamma}^B_{\ell''}), \\
    \text{s.t.} & \quad \gamma^L_{\ell'} + \gamma^B_{\ell''} \leq z, \\
    & \quad 0 \leq \gamma^L_{\ell'}, \gamma^B_{\ell''} \leq 1.
\end{align*}
\]

Because Model (17) is a linear program, it has an extreme point optimal solution. Plotting the feasible region (17b)-(17c) reveals that the extreme points are $(\gamma^L_{\ell'}, \gamma^B_{\ell''}) \in \{(0, 0), (0, z), (z, 0)\}$ if $z \leq 1$ and $(\gamma^L_{\ell'}, \gamma^B_{\ell''}) \in \{(0, 0), (0, 1), (1, 0), (z - 1, 1), (1, z - 1)\}$ if $z > 1$. Noting that all extreme points have either $\gamma^L_{\ell'}$ or $\gamma^B_{\ell''}$ as an integer, solving (17) yields a solution $(\hat{\gamma}^L, \hat{\gamma}^B) \in U_\Phi$—by replacing $\hat{\gamma}^L_{\ell'}$ and $\hat{\gamma}^B_{\ell''}$ in $(\hat{\gamma}^L, \hat{\gamma}^B)$ with the obtained LP solution—such that $F(\hat{\gamma}^L, \hat{\gamma}^B) \leq F(\hat{\gamma}^L, \hat{\gamma}^B)$ with at most one of $\{\hat{\gamma}^L_{\ell'}, \hat{\gamma}^B_{\ell''}\}$ noninteger. Because this solution is feasible to Model (13) and its objective value improves upon $(\hat{\gamma}^L, \hat{\gamma}^B)$, this solution must be optimal to Model (13).

Case 2: Suppose $\hat{\gamma}^L_{\ell'}$ and $\hat{\gamma}^B_{\ell''}$ are fractional. Define $z = \hat{\gamma}^L_{\ell'} + \hat{\gamma}^B_{\ell''}$. In Model (13), when
in case 1, this implies existence of \( \tilde{\gamma}_\ell \), \( \ell \in \{1, \ldots, N\} \setminus \{\ell'\} \) and \( \gamma_\ell^B = \hat{\gamma}_\ell^B \), \( \ell \in \{1, \ldots, N\} \setminus \{\ell''\} \) are fixed, the model reduces to

\[
\min f_\ell(\gamma_\ell^L, \tilde{\gamma}_\ell^B) + f_{\ell''}(\tilde{\gamma}_{\ell''}^L, \hat{\gamma}_{\ell''}^B),
\]
\[
s.t. \quad \gamma_\ell^L + \gamma_{\ell''}^B \leq z, \quad 0 \leq \gamma_\ell^L, \hat{\gamma}_{\ell''}^B \leq 1,
\]
but this also has an extreme point optimal solution (see, e.g., classical nonlinear programming references [2, 33]) because \( f_\ell(\gamma_\ell^L, \tilde{\gamma}_\ell^B) \) is linear in \( \gamma_\ell^L \) and \( f_{\ell''}(\tilde{\gamma}_{\ell''}^L, \hat{\gamma}_{\ell''}^B) \) is concave in \( \gamma_{\ell''}^B \) (due to Theorem 1), thus implying objective (18a) is concave. By the same argument as in case 1, we can obtain \( (\hat{\gamma}_L, \hat{\gamma}_B) \) that is optimal for Model (13) such that at most one of \{\hat{\gamma}_L^L, \hat{\gamma}_B^B\} is noninteger.

Case 3: Suppose \( \tilde{\gamma}_L^B \) and \( \tilde{\gamma}_{\ell''}^B \) are fractional and let \( z = \tilde{\gamma}_L^B + \tilde{\gamma}_{\ell''}^B \). When \( \gamma_\ell^L = \hat{\gamma}_L^L \) and \( \gamma_\ell^B = \tilde{\gamma}_L^B \), \( \ell \in \{1, \ldots, N\} \setminus \{\ell', \ell''\} \) are fixed in Model (13), the model becomes

\[
\min f_\ell(\tilde{\gamma}_L^L, \gamma_\ell^B) + f_{\ell''}(\tilde{\gamma}_{\ell''}^L, \hat{\gamma}_{\ell''}^B),
\]
\[
s.t. \quad \gamma_{\ell''}^B + \hat{\gamma}_{\ell''}^B \leq z, \quad 0 \leq \tilde{\gamma}_L^L, \hat{\gamma}_{\ell''}^B \leq 1.
\]

According to Theorem 1, \( f_\ell(\tilde{\gamma}_L^L, \gamma_\ell^B) \) and \( f_{\ell''}(\tilde{\gamma}_{\ell''}^L, \hat{\gamma}_{\ell''}^B) \) are concave in \( \gamma_\ell^B \) and \( \gamma_{\ell''}^B \), respectively. Therefore, Model (19) has an extreme point optimal solution. Analogous to the argument given in case 1, this implies existence of \( (\hat{\gamma}_L^L, \hat{\gamma}_B^B) \) that is optimal for Model (13) with at most one of \{\hat{\gamma}_L^L, \hat{\gamma}_B^B\} noninteger.

Note that in the case where both \{\gamma_\ell^L, \gamma_\ell^B\} are noninteger for \( \ell = \ell' \) and/or \( \ell = \ell'' \), we may apply cases 1–3 at most three times to construct an optimal solution to Model (13) with only \( k^* - 1 \) fractional subsystems. This establishes a contradiction (as \( (\hat{\gamma}_L^L, \hat{\gamma}_B^B) \) was selected with minimum number of fractional subsystems) and proves \( k^* \leq 1 \).

\[\square\]

**Corollary 1.** There exists an optimal solution of Model (13) such that \( \gamma_\ell^L + \gamma_\ell^B \in \{0, \Phi - [\Phi], 1, 1 + \Phi - [\Phi], 2\}, \forall \ell = 1, \ldots, N. \)

**Proof.** In Remark 1, it is proven that \( F(\gamma_1^L, \gamma_1^B) \) is decreasing in \( \gamma_1^L \) and \( \gamma_1^B \), \( \ell = 1, \ldots, N \) whenever all other \( \gamma^L\)- and \( \gamma^B\)-variables are fixed; hence in an optimal solution, \( \sum_{\ell=1}^N (\gamma_\ell^L + \gamma_\ell^B) = \Phi \). Due to Theorem 2, there exists an optimal solution in which there are \( N - 1 \) subsystems \( \ell \) such that \( \gamma_\ell^L \) and \( \gamma_\ell^B \) are integer. Without loss of generality, suppose this is the case for subsystems \( 1, \ldots, N - 1 \). Because \( \Phi = \sum_{\ell=1}^N (\gamma_\ell^L + \gamma_\ell^B) \) and \( \gamma_\ell^L + \gamma_\ell^B \) is integer for \( \ell = 1, \ldots, N - 1 \), subsystem \( N \) satisfies

\[
(\gamma_N^L + \gamma_N^B) - [\gamma_N^L + \gamma_N^B] = \Phi - [\Phi],
\]
which yields the result. \[\square\]
Remark 2. If $\Phi$ is integer, Corollary 1 implies existence of an optimal solution of Model (13) in which $\gamma_\ell^L + \gamma_\ell^B \in \{0, 1, 2\}$, $\forall \ell = 1, \ldots, N$.

In the following, we use the previous result and local optimality conditions to further restrict the candidate set of optimal solutions to the separation problem such that the final candidate set is finite for given $\tau$, giving rise to the dynamic programming approach for solving the problem.

**Theorem 3.** For $0 < z < 2$ and $\tau_\ell > 1$, define quadratic equation coefficients $b_\ell(z)$ and $c_\ell(z)$ as

\[ b_\ell(z) = -\left[2\sigma_\ell^L + (\bar{\beta}_\ell \sigma_\ell^L + \sigma_\ell^B (-\bar{\lambda}_\ell + z\sigma_\ell^L)) \ln(\tau_\ell)\right] \sigma_\ell^B, \]  
(21a)

\[ c_\ell(z) = -\left[\sigma_\ell^B (\bar{\lambda}_\ell - z\sigma_\ell^L) - \bar{\beta}_\ell \sigma_\ell^L + \sigma_\ell^B \ln(\tau_\ell) \bar{\lambda}_\ell (z\sigma_\ell^B + \bar{\beta}_\ell)\right], \]  
(21b)

and define $a_\ell = (\sigma_\ell^B)^2 \sigma_\ell^L \ln(\tau_\ell)$. In this case

\[
\min \{ f_\ell (\gamma_\ell^L, \gamma_\ell^B) \mid \gamma_\ell^L + \gamma_\ell^B = z, 0 \leq \gamma_\ell^L, \gamma_\ell^B \leq 1\},
\]  
(22)

has an optimal solution in the set

\[
C(z, \tau_\ell) = \left\{(\lfloor z \rfloor, z - \lfloor z \rfloor), \left(\frac{-b_\ell(z) \pm \sqrt{b_\ell(z)^2 - 4a_\ell c_\ell(z)}}{2a_\ell}, z - \frac{-b_\ell(z) \pm \sqrt{b_\ell(z)^2 - 4a_\ell c_\ell(z)}}{2a_\ell}\right), \left(\frac{-b_\ell(z) \mp \sqrt{b_\ell(z)^2 - 4a_\ell c_\ell(z)}}{2a_\ell}, z - \frac{-b_\ell(z) \mp \sqrt{b_\ell(z)^2 - 4a_\ell c_\ell(z)}}{2a_\ell}\right), (z - \lfloor z \rfloor, \lfloor z \rfloor)\right\}.
\]  
(23)

**Proof.** When $\gamma_\ell^L + \gamma_\ell^B = z$, Model (22) simplifies (because $f_\ell$ is decreasing) for subsystem $\ell$ by making the substitution $\gamma_\ell^B = z - \gamma_\ell^L$ to obtain the one-variable model

\[
\min g_\ell (z, \gamma_\ell^L) = f_\ell (\gamma_\ell^L, z - \gamma_\ell^L),
\]  
(24a)

s.t. \[ \max\{0, z - 1\} \leq \gamma_\ell^L \leq \max\{1, z\}. \]  
(24b)

The minimum value of $g_\ell (z, \gamma_\ell^L)$ over $\max\{0, z - 1\} \leq \gamma_\ell^L \leq \max\{1, z\}$ occurs either at its extreme points $\gamma_\ell^L \in \{\lfloor z \rfloor, z - \lfloor z \rfloor\}$ or when $\partial g_\ell (z, \gamma_\ell^L) / \partial \gamma_\ell^L = 0$. Letting $K = -\tau_\ell^{z\sigma_\ell^B + \bar{\beta}_\ell - \sigma_\ell^B \gamma_\ell^L - 1}$, the derivative is given as

\[
\frac{\partial g_\ell (z, \gamma_\ell^L)}{\partial \gamma_\ell^L} = K \left(\bar{\beta}_\ell \sigma_\ell^L - \sigma_\ell^B (\bar{\lambda}_\ell + \sigma_\ell^L (2\gamma_\ell^L - z)) - \sigma_\ell^B (\sigma_\ell^B (z - \gamma_\ell^L) + \bar{\beta}_\ell)(\bar{\lambda}_\ell + \gamma_\ell^L \sigma_\ell^L) \ln(\tau_\ell)\right),
\]  
(25)

which yields the roots $\gamma_\ell^L = \frac{-b_\ell(z) \pm \sqrt{b_\ell(z)^2 - 4a_\ell c_\ell(z)}}{2a_\ell}$ under $\partial g_\ell (z, \gamma_\ell^L) / \partial \gamma_\ell^L = 0$. \[ \square \]

**Remark 3.** For $\tau_\ell = 1$, the root of $\partial g_\ell (z, \gamma_\ell^L) / \partial \gamma_\ell^L = 0$ is $\gamma_\ell^L = \frac{\bar{\beta}_\ell \sigma_\ell^L + \sigma_\ell^B (z\sigma_\ell^L + \bar{\lambda}_\ell)}{2\sigma_\ell^L}$. Therefore, in
this case, Model (22) has an optimal solution in
\[
C(z, 1) = \left\{ \left( [z], z - [z] \right), (z - [z], [z]), \left( \frac{\beta_i \sigma^L \ell + \sigma^P \ell \left( \sigma^L \ell - \bar{\lambda} \ell \right)}{2\sigma^L \ell}, z - \frac{\beta_i \sigma^L \ell + \sigma^P \ell \left( \sigma^L \ell - \bar{\lambda} \ell \right)}{2\sigma^L \ell} \right) \right\}.
\] (26)

**Remark 4.** Some of the points in \( C(z, \tau_\ell) \) may be infeasible because they may be negative or greater than one, so the set can be further restricted.

Throughout the remainder of the document, let \( (\gamma^L_\ell(z), \gamma^B_\ell(z)) \) denote any element of the set \( \arg\min \{f_\ell(\gamma^L_\ell, \gamma^B_\ell) \mid (\gamma^L_\ell, \gamma^B_\ell) \in C(z, \tau_\ell); 0 \leq \gamma^L_\ell, \gamma^B_\ell \leq 1 \} \). By Theorem 3 and Remark 4, \( (\gamma^L_\ell(z), \gamma^B_\ell(z)) \) provides an optimal solution to Model (22). In Theorems 4 and 5, we extend these results to summarize candidate solutions for Model (13).

**Theorem 4.** For integer value \( \Phi \), define
\[
S_{\ell, \Phi} \equiv \{(\gamma^L_\ell, \gamma^B_\ell) \in U_\Phi \mid (\gamma^L_\ell, \gamma^B_\ell) \in \{(0, 0), (\gamma^L_\ell(1), \gamma^B_\ell(1)), (1, 1)\}\},
\] (27)
and let \( S_\Phi \equiv \bigcap_{\ell=1}^N S_{\ell, \Phi} \). For integer-valued \( \Phi \), the set \( S_\Phi \) contains an optimal solution to Model (13).

**Proof.** Due to Remark 2, there is an optimal solution of Model (13) that satisfies \( \gamma^L_\ell + \gamma^B_\ell \in \{0, 1, 2\} \) for all subsystems \( \ell = 1, \ldots, N \). When \( \gamma^L_\ell + \gamma^B_\ell = 0 \), we have \( \gamma^L_\ell = \gamma^B_\ell = 0 \), and when \( \gamma^L_\ell + \gamma^B_\ell = 1 \), we have \( \gamma^L_\ell = \gamma^B_\ell = 1 \). Also, \( (\gamma^L_\ell(1), \gamma^B_\ell(1)) \) is optimal for Model (13) when \( \gamma^L_\ell + \gamma^B_\ell = 1 \). \( \Box \)

For noninteger \( \Phi \), we also show that there exists an optimal solution to Model (13) among a finite set of candidate solutions in Theorem 5.

**Theorem 5.** For noninteger \( \Phi \), define \( \alpha = \Phi - \lfloor \Phi \rfloor \) and \( \alpha' = 1 + \Phi - \lfloor \Phi \rfloor \). Define
\[
S_{\ell, \Phi} \equiv \{(\gamma^L_\ell, \gamma^B_\ell) \in U_\Phi \mid (\gamma^L_\ell, \gamma^B_\ell) \in \{(0, 0), (\gamma^L_\ell(\alpha), \gamma^B_\ell(\alpha)), (\gamma^L_\ell(1), \gamma^B_\ell(1)), (\gamma^L_\ell(\alpha'), \gamma^B_\ell(\alpha'))\} \},
\] (28)
and let \( S_\Phi \equiv \bigcap_{\ell=1}^N S_{\ell, \Phi} \). When \( \Phi \) is noninteger, \( S_\Phi \) contains an optimal solution to Model (13).

**Proof.** According to Corollary 1, there is an optimal solution of Model (13) satisfying \( \gamma^L_\ell + \gamma^B_\ell \in \{0, \Phi - \lfloor \Phi \rfloor, 1, 1 + \Phi - \lfloor \Phi \rfloor, 2\} \) for all subsystems \( \ell = 1, \ldots, N \). Hence, based on Theorem 3, there exists an optimal solution to Model (13) in which
\[
(\gamma^L_\ell, \gamma^B_\ell) \in \{(0, 0), (\gamma^L_\ell(\alpha), \gamma^B_\ell(\alpha)), (\gamma^L_\ell(1), \gamma^B_\ell(1)), (\gamma^L_\ell(\alpha'), \gamma^B_\ell(\alpha'))\}, (1, 1)\}.
\] (29)
\( \Box \)
Theorems 4 and 5 establish a finite candidate set of optimal solutions for Model (13). This enables an efficient dynamic programming approach for solving Model (13), in which the number of stages is equal to the number of subsystems, and each stage $\ell$ contains a number of possible states corresponding to the amount of the original uncertainty budget that remains at the beginning of that stage. We now describe the dynamic programming algorithm.

For integer $\Phi$, define $Q_\Phi = \{1, \ldots, \Phi\}$ and for noninteger $\Phi$ define $Q_\Phi = \{1, \ldots, \lfloor \Phi \rfloor\} \cup \{\lfloor \Phi \rfloor, 1 + (\Phi - \lfloor \Phi \rfloor), \ldots, \lfloor \Phi \rfloor + (\Phi - \lfloor \Phi \rfloor) = \Phi\}$. Initialize the value function as $q_0(\phi) = 0$ for all $\phi \in Q_\Phi$. For $\ell = 1, \ldots, N$ and integer $\phi \in \{Q_\Phi \mid \phi \in \mathbb{Z}\}$, calculate $q_\ell(\phi)$ as

$$q_\ell(\phi) = \min\{q_{\ell-1}(\phi) + f_\ell(0,0), q_{\ell-1}(\phi - 1) + f_\ell(\bar{\gamma}_\ell^L(1), \bar{\gamma}_\ell^B(1)), q_{\ell-1}(\phi - 2) + f_\ell(1,1)\}. \quad (30)$$

For $\ell = 1, \ldots, N$ and noninteger $\phi \in \{Q_\Phi \mid \phi \notin \mathbb{Z}\}$, calculate $q_\ell(\phi)$ as

$$q_\ell(\phi) = \min\{q_{\ell-1}(\phi) + f_\ell(0,0), q_{\ell-1}(\phi - \alpha) + f_\ell(\bar{\gamma}_\ell^L(\alpha), \bar{\gamma}_\ell^B(\alpha)), q_{\ell-1}(\phi - 1) + f_\ell(\bar{\gamma}_\ell^L(1), \bar{\gamma}_\ell^B(1)), q_{\ell-1}(\phi - 2) + f_\ell(1,1)\}, \quad (31)$$

in which $\alpha$ and $\alpha'$ are defined in Theorem 5. The optimal objective value of Model (13) is equal to $G(\tau) = q_N(\Phi)$.

We now describe an exact solution approach for Model (11). By applying the variable substitution $L \equiv \ln R$ to the equivalent Model (12), Model (11) is equivalent (due to monotonicity of the natural log function) to

$$\max \quad L,$$  

s.t.  

$$\sum_{\ell=1}^{N} - (\tilde{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\bar{\beta}^L_\ell + \sigma^B_\ell \gamma^B_\ell)T_\ell^{(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 1)} t \geq L, \quad \forall (\gamma^L, \gamma^B) \in U_\Phi, \quad (32a)$$  

$$\tau \in X. \quad (32b)$$

Replacing $U_\Phi$ in (32b) with a subset $\bar{D} \subset U_\Phi$ yields the outer approximation

$$E(\bar{D}) = \max \quad L,$$  

s.t.  

$$\sum_{\ell=1}^{N} - (\tilde{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell)T_\ell^{(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 1)} t \geq L, \quad \forall (\gamma^L, \gamma^B) \in \bar{D}, \quad (33a)$$  

$$\tau \in X. \quad (33b)$$

Because the feasible region of Model (33) contains the feasible region of Model (32), Model (33) provides an upper bound for the logarithm of the objective value of Model (11) (i.e., $e^{E(\bar{D})} \geq R^*$).

Algorithm 1 formally states the cutting-plane algorithm.
Algorithm 1 Cutting plane (calculate $R^*$ within $\epsilon$ tolerance).

1: function CUTTINGPLANE ($\Phi$, $\epsilon$)
2: Initialize $\bar{D} \subset U\Phi$. Set $LB \leftarrow 0$ and $UB \leftarrow 1$.
3: Solve the outer approximation problem with optimal objective $e^{E(D)}$ and solution $\hat{\tau}$.
4: Set $UB \leftarrow e^{E(D)}$.
5: Solve the dynamic programming with optimal objective $e^{G(\hat{\tau})}$ and solution $(\hat{\tau}^L, \hat{\tau}^B)$.
6: If $e^{G(\hat{\tau})} > LB$ then set $LB \leftarrow e^{G(\hat{\tau})}$ and $\tau^* \leftarrow \hat{\tau}$.
7: If $UB - LB < \epsilon$ then return $\tau^*$ \Comment{\tau^* is $\epsilon$-optimal}
8: Set $\bar{D} = \bar{D} \cup (\hat{\tau}^L, \hat{\tau}^B)$. Go to Line 3.
9: end function

In Theorem 6, we prove that Model (33) is a convex optimization problem, so the cutting-plane algorithm can be solved efficiently using general purpose convex optimization software.

Theorem 6. Model (33) is a convex optimization problem.

Proof. For fixed value $(\gamma^L, \gamma^B)$, define $h_\ell(\tau_\ell)$ such that

$$h_\ell(\tau_\ell) = -(\bar{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell)\tau_\ell^{(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 1)} t. \quad (34)$$

The first and second derivatives of $h_\ell(\tau_\ell)$ with respect to the parameter $\tau_\ell$ are equal to

$$\frac{\partial h_\ell(\tau_\ell)}{\partial \tau_\ell} = -(\bar{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 1)\tau_\ell^{(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 2)} t, \quad (35a)$$

$$\frac{\partial^2 h_\ell(\tau_\ell)}{\partial^2 \tau_\ell} = -(\bar{\lambda}_\ell + \sigma^L_\ell \gamma^L_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 1)(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 2)\tau_\ell^{(\bar{\beta}_\ell + \sigma^B_\ell \gamma^B_\ell - 3)} t < 0. \quad (35b)$$

Therefore, $h_\ell(\tau_\ell)$ is concave in $\tau_\ell$. Constraint (33b) is equivalent to $\sum_{\ell=1}^{N} h_\ell(\tau_\ell) \geq L$. Because each $h_\ell(\tau_\ell)$-term is concave, $\sum_{\ell=1}^{N} h_\ell(\tau_\ell)$ is concave and (33b) thus define a convex region. The remaining constraints are linear; hence (33) has a convex feasible region and a linear objective and is therefore a convex optimization problem. \hfill \square

We now prove the convergence of the algorithm.

Lemma 1. Define $\tau^\ell_{\min} \equiv \tau^\ell_0$ and $\tau^\ell_{\max} \equiv \min_{u=1}^{V} \left\{ \frac{(b_u - \sum_{\ell=1}^{N} c^L_\ell \tau^L_\ell + c^B_\ell \tau^B_\ell)}{e^L_\ell} \right\}$ as the minimum and maximum possible testing time for subsystem $\ell = 1, \ldots, N$, respectively. For $\delta > 0$, let $K_\ell = \left\lceil \frac{\tau^\max_\ell - \tau^\min_\ell}{\delta} \right\rceil$ and $K = \max_{\ell=1}^{N} \{ K_\ell \}$. Define $\tau^k_\ell$, $\ell = 1, \ldots, N$ as the obtained optimal solution of the outer approximation in iteration $k$. Among $K^N + 1$ iterations, there are two iterations $k'$ and $k''$ such that $|\tau^{k''}_\ell - \tau^{k'}_\ell| \leq \delta$ for all $\ell = 1, \ldots, N$.

Proof. For each subsystem $\ell = 1, \ldots, N$, divide interval $[\tau^\min_\ell, \tau^\max_\ell]$ to $K$ equal and disjoint intervals with lengths equal to $\left( \frac{\tau^\max_\ell - \tau^\min_\ell}{K} \right) / K_\ell \leq \delta$. Each subsystem’s testing time is in one of $K$ intervals, hence there are no more than $K^N$ combinations of possible intervals for system testing.
time solution $\tau^k$. Therefore, after $K^N + 1$ iterations, there are two solutions with testing times in same intervals.

Lemma 2. For a series-parallel system, consider $\bar{N}$ and $\bar{M}_\ell$ as the number of subsystems and number of components in subsystem $\ell = 1, \ldots, \bar{N}$, respectively. Define subsystem reliabilities $\bar{\rho}_{\ell,i}$ and $\bar{\rho}_{\ell,i}$ such that $0 \leq \bar{\rho}_{\ell,i} \leq \bar{\rho}_{\ell,i} \leq 1$, $\ell = 1, \ldots, \bar{N}$, $i = 1, \ldots, M_\ell$. Also, consider $\bar{\epsilon}_{\ell,i} = \bar{\rho}_{\ell,i} - \bar{\rho}_{\ell,i}$, $\bar{\epsilon}_{\ell,i} = \max_{i=1,\ldots,\bar{N}} \{\epsilon_{\ell,i}\}$, $\bar{\epsilon}_{\ell,i} = \max_{i=1,\ldots,\bar{M}_\ell} \{\bar{M}_\ell\}$, and $\bar{\epsilon} = \max_{N=1} [1 - (N+1)M_{\text{max}} / \epsilon_{\text{max}}]$. If $\bar{M}_{\text{max}} \bar{\epsilon}_{\text{max}} \leq 1$ and $N\bar{M}_{\text{max}} \bar{\epsilon}_{\text{max}} / (1 - (N+1)M_{\text{max}}) \leq 1$, then

$$\prod_{\ell=1}^{\bar{N}} \left[ 1 - \prod_{i=1}^{\bar{M}_\ell} (1 - \bar{\rho}_{\ell,i}) \right] - \prod_{\ell=1}^{\bar{N}} \left[ 1 - \prod_{i=1}^{\bar{M}_\ell} (1 - \bar{\rho}_{\ell,i}) \right] \leq \bar{\epsilon}.$$

(36)

Proof. This is a lemma from [23].

Lemma 3. For solutions $\tau^{k''}$ and $\tau^{k'}$, define $\epsilon_{\ell} = |R(\tau^{k''}_\ell; t, \lambda, \beta) - R(\tau^{k'}_\ell; t, \lambda, \beta)|$, $\epsilon_{\text{max}} = \max_{\ell=1,\ldots,N} \{\epsilon_{\ell}\}$, and $\epsilon \equiv N\epsilon_{\text{max}} / [1 - (N+1)\epsilon_{\text{max}}]$. If $\epsilon_{\text{max}} \leq 1 / (N+1)$, then $|R(\tau^{k''}; \lambda, \beta) - R(\tau^{k'}; \lambda, \beta)| \leq \epsilon$.

Proof. We invoke a special case of Lemma 2. We consider a series system, so $\bar{M}_\ell = 1$. Define $\epsilon_{\ell}$ as the absolute value of the difference of $|R(\tau^{k'}_\ell; t, \lambda, \beta) - R(\tau^{k''}_\ell; t, \lambda, \beta)|$. By setting $N = \bar{N}$, $M_{\text{max}} = 1$, $\epsilon_{\text{max}} = \epsilon_{\text{max}}$ and $\epsilon = \bar{\epsilon}$ in Lemma 2, $\bar{M}_{\text{max}} \epsilon_{\text{max}} \leq 1$ and $N\bar{M}_{\text{max}} \epsilon_{\text{max}} / (1 - (N+1)\epsilon_{\text{max}}) \leq 1$ result in $\epsilon_{\text{max}} \leq 1 / (N+1)$, $\bar{\epsilon} \equiv N\epsilon_{\text{max}} / [1 - (N+1)\epsilon_{\text{max}}]$. Under $\bar{\rho}_{\ell,1} = \max \{ R(\tau^{k'}_\ell; t, \lambda, \beta), R(\tau^{k''}_\ell; t, \lambda, \beta) \}$ and $\bar{\rho}_{\ell,1} = \min \{ R(\tau^{k'}_\ell; t, \lambda, \beta), R(\tau^{k''}_\ell; t, \lambda, \beta) \}$, Lemma 2 implies

$$\prod_{\ell=1}^{\bar{N}} R(\tau^{k''}_\ell; t, \lambda, \beta) - \prod_{\ell=1}^{\bar{N}} R(\tau^{k'}_\ell; t, \lambda, \beta) \leq \prod_{\ell=1}^{\bar{N}} \bar{\rho}_{\ell,1} - \prod_{\ell=1}^{\bar{N}} \bar{\rho}_{\ell,1} = \prod_{\ell=1}^{\bar{N}} \bar{\rho}_{\ell,1} - \prod_{\ell=1}^{\bar{N}} \bar{\rho}_{\ell,1} \leq \epsilon.$$

(37)

Theorem 7. For a series system, the cutting-plane algorithm identifies an $\epsilon$-optimal solution in a finite number of iterations.

Proof. In [23], it is proven that for a given $\epsilon_{\text{max}} > 0$, there exists $\delta_{\ell} > 0$ such that $R(\tau^{k'}_\ell; t, \lambda, \beta) - R(\tau^{k''}_\ell; t, \lambda, \beta) \leq \epsilon_{\text{max}}$ for $\tau^{k''}_\ell \leq \tau_\ell \leq T_\ell$. Define $\delta = \min_{\ell=1,\ldots,N} \{\delta_{\ell}\}$. According to Lemma 1, there exist two iterations $k'$ and $k''$ such that $|\tau^{k''}_\ell - \tau^{k'}_\ell| \leq \delta$, which suggests $R(\tau^{k''}_\ell; t, \lambda, \beta) - R(\tau^{k'}_\ell; t, \lambda, \beta) \leq \epsilon_{\text{max}}$. Set $\epsilon_{\text{max}} = \epsilon / [N + \epsilon(N + 1)]$, therefore, $\epsilon_{\text{max}} \leq [1 / N + 1]$. However,

$$\epsilon_{\text{max}} = \frac{\epsilon}{[N + \epsilon(N + 1)]} \leq \frac{\epsilon}{[\epsilon(N + 1)]} = \frac{1}{[N + 1]}.$$

(38)

By applying Lemma 3, $|R(\tau^{k''}; \lambda, \beta) - R(\tau^{k'}; \lambda, \beta)| \leq \epsilon$. Without loss of generality, assume $k' < k''$, and consider $(\bar{\lambda}, \bar{\beta}) = (\bar{\lambda} + \sigma^L \hat{\gamma}^L, \bar{\beta} + \sigma^R \hat{\gamma}^R)$ as the optimal solution of the separation problem.
in iteration $k'$, so $R(\tau^{k'}, \hat{\lambda}, \hat{\beta})$ is a lower bound for $R^*$ (the optimal objective value of Model (11)) in iteration $k'$. Because $k' < k''$, $(\hat{\gamma}^L, \hat{\gamma}^B) \in \tilde{D}$ in iteration $k''$, so

$$R(\tau^{k''}, \hat{\lambda}, \hat{\beta}) \geq e^{E(D)} \geq R^*, \tag{39}$$

which suggests $R(\tau^{k''}, \hat{\lambda}, \hat{\beta})$ is an upper bound for $R^*$.

In the following section, we develop the robust optimization models and solution methodology for a series-parallel system. However, before doing so, we first provide some commentary regarding the assumption that $\tau_0^\ell \geq 1$, $\ell = 1, \ldots, N$, that was used in this section. This assumption only enables solving the separation model, Model (13), via dynamic programming. Given some other algorithm for solving Model (13), the outer approximation Model (33) remains a convex optimization problem. The convergence result, Theorem 7, extends in this case as well, although the required computational effort per iteration may be increased as a result of having to solve Model (13) by another means. The series-parallel algorithm developed in the following section is somewhat more computationally complex than its series-system counterpart; however, it does not require the assumption that $\tau_0^\ell \geq 1$, $\ell = 1, \ldots, N$. As a result, one alternative for solving Model (13) is the special case of Algorithm 2—the series-parallel separation algorithm provided in the following section—specialized to the case of one component per subsystem.

4 Solution Method for Series-Parallel Systems

In this section, we develop a model for the robust allocation of testing resources across the subsystems of a series-parallel system, in which the components inside each subsystem are identical. In this system, there are $N$ subsystems connected in a series, and each subsystem $\ell = 1, \ldots, N$ includes $M_\ell$ identical components connected in parallel. Based on Equation (5), the robust optimization problem for a series-parallel system is defined as

$$R^* = \max R,$$

s.t. \hspace{1cm} $\prod_{\ell=1}^N \left[1 - \left(1 - e^{-\left(\lambda_\ell + \sigma_L^\ell \gamma_L^\ell \right)\left(\beta_\ell + \sigma_B^\ell \gamma_B^\ell \right)^\ell M_\ell \right)^{M_\ell} \right] \geq R, \forall (\gamma^L, \gamma^B) \in U_\Phi, \tag{40a}$$

$$\tau \in X, \tag{40b}$$

in which $U_\Phi$ is defined in Equation (8).

Model (40) is nonlinear and nonconvex, so there is no apparent exact solution method. We now describe such a method based on a cutting-plane algorithm. The separation model associated with
calculating the robust objective value for \( \tau \in X \) is given by
\[
G(\tau) = \min \prod_{\ell=1}^{N} \left[ 1 - \left( 1 - e^{-\left( \beta_{\ell} + \sigma_{\ell}^L \gamma_{\ell}^L \right) \left( \beta_{\ell} + \sigma_{\ell}^B \gamma_{\ell}^B \right) r_{\ell}^p} \right)^{M_{\ell}} \right],
\]
(41a)
s.t. \( (\gamma^L, \gamma^B) \in U_{\Phi} \).
(41b)

A solution of Model (41) is feasible for Model (40); therefore, Model (41) yields a lower bound for Model (40) (i.e. \( G(\tau) \leq R^* \)). Unfortunately, the series-parallel extensions of Theorems 4–5 no longer hold for Model (41). We address this issue by using a discretize-and-refine method developed for the deterministic version of our problem by Heydari et al. [23] (and later adapted for the model that integrates redundancy allocation decisions in [22]). We will adapt this method to solve the separation and outer approximation problems for series-parallel systems. This method is based on the idea that lower and upper bounds on the robust reliability solution can be found by solving the discretized model. Then, by successive refinement of the discretized models, the lower and upper bounds converge to an arbitrarily small gap. We begin by discretizing the separation model. For this purpose, for each subsystem \( \ell = 1, \ldots, N \), consider \( P_{\ell} \) discrete points \( z_{p}^p, p = 1, \ldots, P_{\ell} \), each of which represents a potential value of \( \gamma^L_{\ell} + \gamma^B_{\ell} \), and define \( r_{\ell}^p \) as the reliability associated with point \( p = 1, \ldots, P_{\ell} \). The discretized version of the separation model (41) is represented as
\[
R^{SD}(r, t) = \min \prod_{\ell=1}^{N} \prod_{p=1}^{P_{\ell}} r_{\ell}^p x_{\ell}^p,
\]
(42a)
s.t. \( \sum_{\ell=1}^{N} \sum_{p=1}^{P_{\ell}} z_{p}^p x_{\ell}^p \leq \Phi \),
(42b)
\[
\sum_{p=1}^{P_{\ell}} x_{\ell}^p = 1, \quad \forall \ell = 1, \ldots, N,
\]
(42c)
in which \( r \) is a vector of reliabilities \( r_{\ell}^p \) and
\[
0 \equiv z_1^1 < z_2^1 < \cdots < z_{P_{\ell}}^p \equiv 2(< z_{P_{\ell}+1}^p),
\]
(43)
where \( z_{P_{\ell}+1}^p \) is defined (for notational convenience) to be a constant greater than 2. We now show how Model (42) can provide lower and upper bounds on subsystem reliability. Due to Theorem 3, when \( \gamma^L_{\ell} + \gamma^B_{\ell} \) is fixed to equal \( z \), Model (41) has an optimal solution in the set \( \arg \min \{ f_{\ell}(\gamma^L_{\ell}, \gamma^B_{\ell}) | (\gamma^L_{\ell}, \gamma^B_{\ell}) \in C(z, \tau_{\ell}) \} \). Let \( (\gamma^L_{\ell}(z), \gamma^B_{\ell}(z)) \) denote any such solution. Setting
\[
r_{\ell}^p = 1 - \left( 1 - f_{\ell}(\gamma^L_{\ell}(z_{p}^p+1), \gamma^B_{\ell}(z_{p}^p+1)) \right)^{M_{\ell}}, \quad \forall p = 1, \ldots, P, \ell = 1, \ldots, N,
\]
(44)
in Model (42) provides a lower bound on Model (41), but setting
\[ r^p_\ell = 1 - \left(1 - f_\ell(\bar{\gamma}^L_\ell(z^p_\ell), \bar{\gamma}^B_\ell(z^p_\ell))\right)^{M_\ell}, \forall p = 1, \ldots, P, \ell = 1, \ldots, N, \] (45)
yields an upper bound. Towards proving this formally, define LB-SP and UB-SP as the lower- and upper-bound models, respectively, and define \( \hat{R}(z, t) \) as the optimal objective value of Model (41) when the value of \( \gamma^L_\ell + \gamma^B_\ell \) is equal to \( z_\ell \) for \( \ell = 1, \ldots, N \). (Here, \( z \) denotes the collection of possibly noninteger \( z_\ell \)-values across all subsystems.)

**Theorem 8.** Define \( \overline{r}^p_\ell = 1 - \left(1 - f_\ell(\bar{\gamma}^L_\ell(z^{p+1}_\ell), \bar{\gamma}^B_\ell(z^{p+1}_\ell))\right)^{M_\ell} \) for all \( \ell = 1, \ldots, N, \ p = 1, \ldots, P_\ell \), then \( R^{SD}(\overline{r}, t) \leq G(\tau) \) (LB-SP yields a lower bound on the separation problem).

**Proof.** Let \( \bar{\gamma}^L_\ell \) and \( \bar{\gamma}^B_\ell \), \( \ell = 1, \ldots, N \), denote an optimal solution for Model (41). Then, for each subsystem \( \ell \), find the index \( \hat{P}_\ell \in \{1, \ldots, P_\ell\} \) satisfying \( z^{\hat{P}_\ell}_\ell \leq \bar{\gamma}^L_\ell + \bar{\gamma}^B_\ell < z^{\hat{P}_\ell+1}_\ell \), and set \( \bar{\gamma}_\ell = z^{\hat{P}_\ell+1}_\ell \). The solution \( x^{\hat{P}_\ell}_\ell = 1, \ \ell = 1, \ldots, N \) is a feasible solution to UB-SP, with objective value equal to \( \hat{R}(\bar{z}, t) \). According to Theorem 1, \( e^{f_\ell(\bar{\gamma}^L_\ell, \bar{\gamma}^B_\ell)} \) is nonincreasing over either \( \gamma^L_\ell \) or \( \gamma^B_\ell \) when the other is fixed. Thus, the value of \( 1 - \left(1 - e^{f_\ell(\bar{\gamma}^L_\ell, \bar{\gamma}^B_\ell)}\right)^{M_\ell} \) is nonincreasing over \( \gamma^L_\ell \) or \( \gamma^B_\ell \) when the other is fixed, which yields that \( \hat{R}(\bar{z}, t) \) is nonincreasing over the value of \( \gamma^L_\ell + \gamma^B_\ell \). Therefore,
\[ R^{SD}(\overline{r}, t) \leq \hat{R}(\bar{z}, t) \leq \hat{R}(\bar{z}, t) = G(\tau), \] (46)
which completes the proof.

**Theorem 9.** Define \( \overline{r}^p_\ell = 1 - \left(1 - f_\ell(\bar{\gamma}^L_\ell(z^p_\ell), \bar{\gamma}^B_\ell(z^p_\ell))\right)^{M_\ell} \) for all \( \ell = 1, \ldots, N, \ p = 1, \ldots, P_\ell \), then \( R^{SD}(\overline{r}, t) \geq G(\tau) \) (UB-SP yields an upper bound on the separation problem).

**Proof.** By setting \( \overline{r}^p_\ell = 1 - \left(1 - f_\ell(\bar{\gamma}^L_\ell(z^p_\ell), \bar{\gamma}^B_\ell(z^p_\ell))\right)^{M_\ell} \), for each solution \( \bar{x} \) in Model (42), there is a solution \( \gamma^L_\ell = \sum_{p=1}^{P_\ell} \bar{\gamma}^L_\ell(z^p_\ell)\bar{x}^p_\ell \) and \( \gamma^B_\ell = \sum_{p=1}^{P_\ell} \bar{\gamma}^B_\ell(z^p_\ell)\bar{x}^p_\ell \) with the same objective in Model (41).

The separation problem is solved by Algorithm 2, where \( P_{sub} \) denotes the initial number of evenly spaced discrete points (see Line 5) used to populate Model (42) and \( \epsilon_{sub} \) denotes the desired optimality tolerance. Define the \( N \)-vectors of \( (\gamma^L, \gamma^B) \) as the incumbent (upper bound) solution and define \( \hat{P}_\ell, \ell = 1, \ldots, N \), as the index \( p \) such that \( x^p_\ell = 1 \) in the obtained optimal solution to LB-SP. The LB-SP and UB-SP models are refined to generate tighter bounds in the following iteration by defining new points \( z^{\hat{P}_\ell+1}_\ell \) given by
\[ z^{\hat{P}_\ell+1}_\ell = \left(0.5z^{\hat{P}_\ell}_\ell + 0.5z^{\hat{P}_\ell+1}_\ell \right), \] (47)
incrementing \( P_\ell \leftarrow P_\ell + 1 \) (unless \( \hat{P}_\ell = P_\ell \), in which case the issue is addressed by Line 20 via redefining \( z^{\hat{P}_\ell+1}_\ell \) as \( 0.5z^{\hat{P}_\ell}_\ell + 0.5z^{\hat{P}_\ell+1}_\ell \)).
Algorithm 2 Series-parallel separation algorithm (calculate a lower bound for $R^*$ within $\epsilon_{\text{sub}}$ tolerance).

1: function SEPARATIONALGORITHM ($\epsilon_{\text{sub}}$, $P_{\text{sub}}$, $\tau$)
2: for \( \ell = 1, \ldots, N \) do
3:   Set $z_{\ell}^{\text{min}} \leftarrow 0$ and $z_{\ell}^{\text{max}} \leftarrow 2 + 2/(P_{\text{sub}} - 1)$
4:   for $p = 1, \ldots, P_{\text{sub}} + 1$ do
5:     Set $z_{\ell}^{p} \leftarrow z_{\ell}^{\text{min}} + (p - 1)(z_{\ell}^{\text{max}} - z_{\ell}^{\text{min}})/P_{\text{sub}}$
6:   end for
7: end for
8: Set $LB \leftarrow 0$ and $UB \leftarrow 1$
9: for $\ell = 1, \ldots, N$ do
10:   Set $P_\ell \leftarrow P_{\text{sub}}$
11: end for
12: if $UB - LB < \epsilon_{\text{sub}}$ then return $\gamma^L, \gamma^B$ \hspace{1cm} $\triangleright$ ($\gamma^L, \gamma^B$) is $\epsilon$-optimal
13: Solve the LB-SP model with optimal solution $\gamma^*$ and objective value $R^{SD}(\bar{r}, t)$
14: Calculate $\bar{\gamma}^L_\ell, \bar{\gamma}^B_\ell \in \arg\min \{ f(\bar{\gamma}^L_\ell, \bar{\gamma}^B_\ell) \mid (\bar{\gamma}^L_\ell, \bar{\gamma}^B_\ell) \in C(\gamma^*, \tau_\ell) \}, \forall \ell = 1, \ldots, N$, using Theorem 5
15: if $R^{SD}(\bar{r}, t) > LB$ then set $LB \leftarrow R^{SD}(\bar{r}, t)$ \hspace{1cm} $\bar{\gamma}^L \leftarrow \bar{\gamma}^L$ and $\gamma^B \leftarrow \bar{\gamma}^B$
16: Solve the UB-SP model with optimal solution $P_\ell \in \{ 1, \ldots, P_\ell \}, \ell = 1, \ldots, N$, and objective value $R^{SD}(\bar{r}, t)$
17: if $R^{SD}(\bar{r}, t) < UB$ then set $UB \leftarrow R^{SD}(\bar{r}, t)$
18: for $\ell = 1, \ldots, N$ do
19:   if $0.5 \left( z_{\ell}^{P_\ell} + z_{\ell}^{P_\ell + 1} \right) \leq 2$ then add a new point $z_{\ell}^{P_\ell + 1} \equiv 0.5 \left( z_{\ell}^{P_\ell} + z_{\ell}^{P_\ell + 1} \right)$ and set $P_\ell \leftarrow P_\ell + 1$
20:   if $0.5 \left( z_{\ell}^{P_\ell} + z_{\ell}^{P_\ell + 1} \right) > 2$ then set $z_{\ell}^{P_\ell + 1} = 0.5 \left( z_{\ell}^{P_\ell} + z_{\ell}^{P_\ell + 1} \right)$
21: end for
22: for $\ell = 1, \ldots, N$ do
23:   Reumber the points such that $0 \equiv z_{\ell}^1 < z_{\ell}^2 < \cdots < z_{\ell}^{P_\ell} \equiv 2 \left( < z_{\ell}^{P_\ell + 1} \right)$
24: end for
25: Go to Line 12
26: end function

In the following, we describe an approach for solving the series-parallel robust allocation Model (40). Considering $\bar{D} \subseteq U_\Phi$, the outer approximation problem is given as

$$E(\bar{D}) = \max \ R, \tag{48a}$$

$$\text{s.t. } \prod_{\ell=1}^{N} \left[ 1 - \left( 1 - e^{-(\lambda_\ell + \sigma_\ell^L \gamma^L_\ell)(\beta_\ell + \sigma_\ell^B \gamma^B_\ell)\tau_\ell^{(\beta_\ell + \sigma_\ell^B \gamma^B_\ell - 1)} t} \right)^{M_\ell} \right] \geq R, \tag{48b}$$

$$\forall (\gamma^L, \gamma^B) \in \bar{D}, \tag{48c}$$

$$\tau \in X.$$
of Model (48) is given as

$$R^{OA}(r, t) = \max R,$$

(49a)

subject to

$$\prod_{\ell=1}^{N} \prod_{p=1}^{P} r_{\ell,j}^{p} y_{\ell}^{p} \geq R, \forall j = 1, \ldots, |\bar{D}|,$$

(49b)

$$\sum_{p=1}^{P} r_{\ell,j}^{p} y_{\ell}^{p} \in X,$$

(49c)

$$\sum_{p=1}^{P} y_{\ell}^{p} = 1,$$

(49d)

$$y \in \{0, 1\}, \forall p = 1, \ldots, P.$$  

(49e)

in which $r$ is a vector of reliabilities $r_{\ell,j}^{p}$. Applying a logarithm to Objective (49a) and Equation (49b) yields the equivalent linear model

$$W^{OA}(r, t) = \max L,$$

(50a)

subject to

$$\sum_{\ell=1}^{N} \sum_{p=1}^{P} \ln(r_{\ell,j}^{p}) y_{\ell}^{p} \geq L, \forall j = 1, \ldots, |\bar{D}|,$$

(50b)

Constraints (49c)–(49c),

in which $e^{W^{OA}(r,t)}$ is equal to $R^{OA}(r, t)$, the optimal objective value of Model (49). Define $(\lambda_{\ell,j}, \beta_{\ell,j})$ $\ell = 1, \ldots, N$ as the AMSAA model parameters associated with scenario $j = 1, \ldots, |\bar{D}|$. We now prove that setting

$$r_{\ell,j}^{p} = 1 - \left(1 - e^{-\lambda_{\ell,j} \beta_{\ell,j} r_{\ell,j}^{p} (\beta_{\ell,j} - 1) t} \right)^{M_{\ell}}, \forall \ell = 1, \ldots, N, j = 1, \ldots, |\bar{D}|, p = 1, \ldots, P,$$

(51)

provides a lower bound on the optimal objective value $E(\bar{D})$; while, setting

$$r_{\ell,j}^{p} = 1 - \left(1 - e^{-\lambda_{\ell,j} \beta_{\ell,j} r_{\ell,j}^{p+1} (\beta_{\ell,j} - 1) t} \right)^{M_{\ell}}, \forall \ell = 1, \ldots, N, j = 1, \ldots, |\bar{D}|, p = 1, \ldots, P,$$

(52)

yields an upper bound (Define $r_{\ell,j}^{p+1} \equiv \tau_{\ell,j}^{max}, \ell = 1, \ldots, N, j = 1, \ldots, |\bar{D}|$, where $\tau_{\ell,j}^{max} \equiv \min_{i=1}^{V} \left\{ \left( b_{\ell,j} - \sum_{p=1}^{N} c_{\ell,j}^{p} r_{\ell,j}^{p} + b_{\ell,j}^{p} r_{\ell,j}^{p} \right) / c_{\ell,j}^{p} \right\}$. Define LB-OA when Model (49) provides a lower bound on Model (48), and UB-OA when it provides an upper bound on Model (48).

**Theorem 10.** Define $\tilde{r}_{\ell,j}^{p} = 1 - \left(1 - e^{-\lambda_{\ell,j} \beta_{\ell,j} r_{\ell,j}^{p} (\beta_{\ell,j} - 1) t} \right)^{M_{\ell}}$ for all $\ell = 1, \ldots, N, j = 1, \ldots, |\bar{D}|, p = 1, \ldots, P$, then $R^{OA}(r, t) \leq E(\bar{D})$. (That is, LB-OA provides a lower bound for the outer approximation problem).

**Proof.** By setting $\tilde{r}_{\ell,j} = 1 - \left(1 - e^{-\lambda_{\ell,j} \beta_{\ell,j} r_{\ell,j}^{p} (\beta_{\ell,j} - 1) t} \right)^{M_{\ell}}$, for each feasible solution $\tilde{y}$ of Model (49),
there is a solution $\tau_\ell = \sum_{p=1}^{P} \tau_P^p y_P^p$ in Model (48) with the same objective value.

**Theorem 11.** Define $\bar{r}_{\ell,j} = 1 - \left(1 - e^{-\lambda_{\ell,j} \beta_{\ell,j} \tau_P^p (\beta_{\ell,j} - 1)}\right)^{M_{\ell}}$ for all $\ell = 1, \ldots, N$, $j = 1, \ldots, |\bar{D}|$, $p = 1, \ldots, P$, then $R^{OA}(\bar{r}, t) \geq E(\bar{D})$. (That is, UB-OA provides an upper bound for the outer approximation problem).

**Proof.** Let $A(\hat{\tau})$ denote the optimal objective value of Model (48) when $\tau = \hat{\tau}$ is fixed. Define $\tau^*_{\ell}, \ell = 1, \ldots, N$, as an optimal solution for Model (48). Find the index $\hat{P}_\ell \in \{1, \ldots, P\}$ for each subsystem $\ell$ such that $\tau_{\ell,\hat{P}_\ell} \leq \tau^*_{\ell} < \tau_{\ell,\hat{P}_\ell+1}$, and set $\hat{\tau}_\ell = \tau_{\ell,\hat{P}_\ell+1}$. The solution $\hat{y}_{\ell,\hat{P}_\ell} = 1$, $\ell = 1, \ldots, N$ is a feasible solution to UB-OA, with objective value equal to $A(\hat{\tau})$. The value $A(\hat{\tau})$ is no less than $R^* = A(\tau^*)$ since $A(\tau)$ is nondecreasing in $\tau$. Hence, $R^* \leq A(\hat{\tau}) \leq R^{OA}(\bar{r}, t)$. 

Algorithm 3 is used to solve the outer approximation Model (48), where $P_{sub}$ is the initial number of discrete points used to populate Model (49) and $\epsilon_{sub}$ is the desired optimality tolerance. In this algorithm, $\tau^*$ defines the incumbent solution. Given a desired optimality tolerance $\epsilon$, Algorithm 4 provides the resulting cutting-plane algorithm for robust allocation of testing time in a series-parallel system. For the sake of brevity, we have not provided a convergence proof of Algorithm 4, but we now provide a sketch: Arguments similar to those used in Theorem 7 can be used to bound the number of iterations of the cutting-plane algorithm, and the number of steps per iteration can be bounded using arguments similar to those in [23], which establish finite convergence of discretize-and-refine algorithms for a related class of problems.
Algorithm 3 The outer approximation algorithm (calculate an upper bound for $R^*$ within $\epsilon_{\text{sub}}$ tolerance).

1: function OUTERAPPROXIMATION ($\epsilon_{\text{sub}}$, $P_{\text{sub}}$)  
2: for $\ell = 1, \ldots, N$ do 
3: \quad Set $\tau_{\ell}^{\text{min}} \leftarrow \tau_0^\ell$ and $\tau_{\ell}^{\text{max}} \leftarrow \min_{V=1}^{N} \left\{ \left( b_v - \sum_{\ell'=1}^{N} c_{\ell'} v + c_{\ell'}^V \right) / c_v^\ell \right\}$ 
4: \quad for $p = 1, \ldots, P_{\text{sub}}$ do 
5: \quad \quad Set $\tau_p^\ell \leftarrow \tau_{\ell}^{\text{min}} + (p - 1) (\tau_{\ell}^{\text{max}} - \tau_{\ell}^{\text{min}}) / P_{\text{sub}}$ 
6: \quad end for 
7: end for 
8: Set $LB \leftarrow 0$, $UB \leftarrow 1$, and $P \leftarrow P_{\text{sub}}$ 
9: If $UB - LB < \epsilon_{\text{sub}}$ then return $\tau^*$ \hfill $\triangleright \tau^*$ is $\epsilon$-optimal 
10: Solve the LB-OA model with optimal solution $\hat{\tau}$ and objective value $R_{\text{OA}}(\bar{r}, t)$ 
11: If $R_{\text{OA}}(\bar{r}, t) > LB$ then set $LB \leftarrow R_{\text{OA}}(\bar{r}, t)$, $\tau^* \leftarrow \hat{\tau}$ 
12: Solve the UB-OA model with optimal solution $\hat{P}_\ell \in \{1, \ldots, P\}$, $\ell = 1, \ldots, N$ and objective value $R_{\text{OA}}(\bar{r}, t)$ 
13: If $R_{\text{OA}}(\bar{r}) < UB$ then set $UB \leftarrow R_{\text{OA}}(\bar{r}, t)$ 
14: for $\ell = 1, \ldots, N$ do 
15: \quad Add a new point $\tau_{\ell}^{P+1} \equiv 0.5 \left( \tau_{\ell}^{P} + \tau_{\ell}^{P+1} \right)$ 
16: end for 
17: Renumber the points such that $\tau_1^\ell < \tau_2^\ell < \cdots < \tau_{P+1}^\ell$ 
18: set $P \leftarrow P + 1$ 
19: Go to Line 9 
20: end function 

Algorithm 4 Cutting plane (calculate $R^*$ within $\epsilon$ tolerance).

1: function CUTTINGPLANE ($V$, $b_v$, $\Phi$, $\epsilon$, $\epsilon_{\text{sub}}$, $P_{\text{sub}}$)  
2: Initialize $\bar{D} \subset U_{\Phi}$. Set $LB \leftarrow 0$ and $UB \leftarrow 1$. 
3: Solve OUTERAPPROXIMATION($\epsilon_{\text{sub}}$, $P_{\text{sub}}$) with objective ($L_{O}, U_{O}$) and solution $\hat{\tau}$. 
4: Set $UB \leftarrow U_{O}$. 
5: Solve SEPARATIONALGORITHM($\epsilon_{\text{sub}}$, $P_{\text{sub}}$, $\hat{\tau}$) with objective ($L_{S}, U_{S}$) and solution $\left( \hat{\gamma}_L, \hat{\gamma}_B \right)$. 
6: If $L_{S} > LB$ then set $LB \leftarrow L_{S}$ and $\tau^* \leftarrow \hat{\tau}$. 
7: If $UB - LB < \epsilon$ then return $\tau^*$ \hfill $\triangleright \tau^*$ is $\epsilon$-optimal 
8: Set $\bar{D} = \bar{D} \cup \left( \hat{\gamma}_L, \hat{\gamma}_B \right)$. Go to Line 3. 
9: end function
5 Computational Results

This section provides a study of computational results. For series systems, the cutting-plane algorithm is coded in MATLAB and CVX solver is used to solve each convex optimization problem. For series-parallel systems, C++ is used to code the cutting-plane algorithm and CPLEX 12.4 is invoked to solve each individual integer program. For all instances, a server with an Intel core i12 with 2.9 GHz and 12 GB RAM is used. We compare the robust and deterministic approaches via sensitivity analysis for a set of numerical examples in Section 5.1, demonstrate the use of confidence intervals to construct uncertainty sets in Section 5.2, study the effect of the system structure on testing time allocation in Section 5.3, and analyze the computational efficiency of the cutting-plane algorithm in Section 5.4. Although we have neither claimed nor demonstrated that that the robust optimization approach always performs better than available deterministic approaches, Sections 5.1–5.2 demonstrate that the robust approach is favorable for what we believe to be a realistic set of instances.

Based upon preliminary experiments, we identified that smaller values of \( \Phi \) (e.g., \( \Phi \in \{0.3, 0.5\} \) when \( N \leq 6 \)) tend to provide more competitive results from the robust optimization model. We believe this is because the scenarios that result when one or more parameters are allowed to take on their worst-case values are, in some cases, too significant of a deterrent. For instance, a subsystem \( \ell \) with \( \bar{\beta}_\ell + \sigma_B^\ell \) close to 1 has, in the worst scenario, almost no growth at all. This discourages solutions in which subsystem \( \ell \) is tested in any significant amount. Because of this observation, we have utilized small values of \( \Phi \) in Sections 5.1–5.2, where we are demonstrating value of the robust approach against available deterministic approaches.

5.1 Comparing Robust and Deterministic Solutions via Sensitivity Analysis

To demonstrate the value of incorporating uncertainty into the resource allocation model, we begin by describing a procedure that can be used to compare the results of alternative testing strategies under different combinations of realized parameter values. We first illustrate the approach graphically for two-subsystem examples in which only two of the four parameters \( \{\lambda_1, \lambda_2, \beta_1, \beta_2\} \) are subject to uncertainty. Using this approach, we then summarize results for larger systems with more decision variables and parameters subject to uncertainty.

The approach utilized in this system considers a fixed set of uncertainty intervals \( [\bar{\lambda}_\ell, \bar{\lambda}_\ell + \sigma_L^\ell] \) and \( [\bar{\beta}_\ell, \bar{\beta}_\ell + \sigma_B^\ell] \) associated with each subsystem \( \ell = 1, \ldots, N \). Hereafter, we will refer to the Cartesian product of these \( 2N \) intervals as the uncertainty box. We generate testing plans \( \tau^{\Phi=0}, \tau^{\Phi=0.5}, \) and \( \tau^{\Phi=2N} \) by solving the robust optimization model corresponding to \( \Phi \in \{0, 0.5, 2N\} \) over the uncertainty box. (Note: The solutions \( \tau^{\Phi=0} \) and \( \tau^{\Phi=2N} \) are respectively the optimistic and pessimistic solutions obtained by solving the corresponding deterministic model.) We then compare the robust solution \( \tau^{\Phi=0.5} \) to the corresponding deterministic solutions over potentially
realized parameters in the uncertainty box.

5.1.1 Series System with $N = 2$, Uncertainty Only in $\lambda$

Consider a series system that contains two subsystems. Suppose the true $\beta$-values are known with certainty such that $\beta_1 = \bar{\beta}_1 = 0.65$ and $\beta_2 = \bar{\beta}_2 = 0.6$ and $\sigma^R_1 = \sigma^R_2 = 0$. Suppose uncertainty in $\lambda$ is described by $\bar{\lambda}_1 = 0.00015$, $\sigma^L_1 = 0.0002$, $\bar{\lambda}_2 = 0.0003$ and $\sigma^L_2 = 0.0001$, and the remaining parameters are given by $t = 18,250$, $V = 1$, $b^1 = 8760$, $c^1_\ell = 1$ and $\tau^0_\ell = 1$, $\ell \in \{1, 2\}$. For this region, we generated $\tau^{\Phi=0}$, $\tau^{\Phi=0.5}$ and $\tau^{\Phi=4}$. (Note, however, that the robust solutions for $2 \leq \Phi \leq 4$ are equal because there is no uncertainty in $\lambda$.) The testing times are $(\tau^{\Phi=0}_1, \tau^{\Phi=0}_2) = (3860.4, 4899.6)$ for the optimistic solution, $(\tau^{\Phi=0.5}_1, \tau^{\Phi=0.5}_2) = (4671.5, 4088.5)$ for the robust solution, and $(\tau^{\Phi=4}_1, \tau^{\Phi=4}_2) = (4748.7, 4011.3)$ for the pessimistic solution. Using these solutions, we then plotted, in Figure 1(a) the $\lambda$-values in the uncertainty box under which each of the three solutions yields the maximum reliability.

The point $(\lambda_1, \lambda_2) = (0.00015, 0.0003)$ in Figure 1(a), falls in region A, thus indicating that the optimistic model’s solution would yield a better reliability for these realized values of $\lambda_1$ and $\lambda_2$. This is to be expected as the optimization model corresponding to $\Phi = 0$ is deterministic under $\lambda_1 = \bar{\lambda}_1 = 0.00015$ and $\lambda_2 = \bar{\lambda}_2 = 0.0003$. Similarly, it is not surprising that $(\lambda_1, \lambda_2) = (0.00035, 0.0004)$ falls in region C, (indicating that the pessimistic model yields superior performance) because the optimization model corresponding to $\Phi = 4$ is deterministic under the assumption that $\lambda_1 = \bar{\lambda}_1 + \sigma^L_1 = 0.00035$ and $\lambda_2 = \bar{\lambda}_2 + \sigma^L_2 = 0.0004$.

We now summarize the analysis for this example in such a way that can be repeated to determine if the robust optimization model’s solutions are a favorable alternative towards using either the optimistic or pessimistic deterministic optimization models. One measure of interest in comparing solutions is the pairwise hypervolume proportion, the proportion of the uncertainty box in which
the robust solution yields higher reliability than either the optimistic or pessimistic deterministic solution. The pairwise hypervolume proportions for the optimistic and pessimistic solutions are respectively 61.87\% (corresponding to Regions B and C in Figure 1(a)) and 74.54\% (corresponding to regions A and B). In other words, if the \( \lambda \)-values are selected uniformly from the uncertainty box, the robust solution generates higher reliability than the optimistic (pessimistic) solution 61.87\% (74.54\%) of the time.

Figure 1(b) depicts a percentile plot of the reliability differences between the robust solution and the optimistic and pessimistic solutions. We sample 1000 realizations of \( \lambda \) uniformly within the uncertainty box and calculate the reliability associated with optimistic, robust and pessimistic solutions. We then calculate the differences between the robust solution and each of the optimistic and pessimistic solutions, sort the reliability differences, and plot the differences in Figure 1(b)—with the smallest difference on the far left, and the largest difference on the far right. The horizontal intercepts occur for the optimistic and pessimistic comparison (approximately) at (1 − 0.6187) and (1 − 0.7454), respectively corresponding to the pairwise hypervolume proportions detailed in the previous paragraph. In this instance, the robust solution’s reliability may exceed that of the optimistic (pessimistic) solution by as much as 0.00472 (0.00053), but the optimistic (pessimistic) solution stands to exceed the reliability of the robust solution by no more than 0.00358 (0.00023).

5.1.2 Series System with \( N = 2 \), Uncertainty Only in \( \beta \)

We now repeat the analysis of the previous section, but this time in the case where \( \lambda \) is known and \( \beta \) is subject to uncertainty. Specifically, let \( \lambda_1 = \bar{\lambda}_1 = 0.00015 \) and \( \lambda_2 = \bar{\lambda}_2 = 0.0003 \) such that their associated uncertainty intervals have a length of zero. The uncertainty underlying parameters are \( \bar{\beta}_1 = 0.65 \), \( \sigma^B_1 = 0.15 \), \( \bar{\beta}_2 = 0.6 \) and \( \sigma^B_2 = 0.05 \). Similar to the previous instance, we set \( t = 18,250 \), \( V = 1 \), \( b_1 = 8760 \), \( c_1^\ell = 1 \) and \( \tau_\ell^0 = 1 \), \( \ell \in \{1, 2\} \). The testing times are now \((\tau_1^{\Phi=0}, \tau_2^{\Phi=0}) = (3860.4, 4899.6)\) for the optimistic solution, \((\tau_1^{\Phi=0.5}, \tau_2^{\Phi=0.5}) = (4656.5, 4103.3)\) for the robust solution, and \((\tau_1^{\Phi=4}, \tau_2^{\Phi=4}) = (4743.5, 4016.5)\) for the pessimistic solution. Because the underlying parameters of the optimistic solution for this instance and the previous instance are equal, the testing times of both instances for the optimistic solution are also equal. Figure 2(a) specifies the policy that provides the highest reliability when \( \beta \)-values are in the uncertainty box. The pairwise hypervolume proportions for the optimistic and pessimistic solutions are 56.23\% (corresponding to regions B and C) and 75.13\% (corresponding to regions A and B). Similar to the previous instance, the robust solution never yields the smallest reliability when it is compared with both optimistic and pessimistic solutions in the uncertainty box. From Figure 2(b), for realizations of \( \beta \) in the uncertainty box, the robust solution’s reliability exceeds the optimistic (pessimistic) solution’s reliability by as much as 0.00439 (0.00065) while the optimistic (pessimistic) solution may exceed the robust solution in reliability by as much as 0.00398 (0.00029).
Figure 2: (a) Uncertainty box partitioned according to the value of $\Phi$ yielding the maximum reliability; and (b) percentile plot of reliability difference between robust and deterministic solutions for the second instance.

5.1.3 Series System with $N = 2$, Uncertainty in Both $\lambda$ and $\beta$

We now analyze the performance of the robust solution in comparison with the optimistic and pessimistic deterministic solutions when the number of uncertain parameters increases. Figure 3 depicts the approximated percentile plot (now generated based on 10,000 sampled realizations of the uncertain parameters since the uncertainty box has a higher dimension) for a series system with two subsystems with $t = 18,250$, $V = 1$, $b_1 = 8760$, $c_2^\ell = 1$ and $\tau_0^\ell = 1$, $\ell \in \{1, 2\}$. None of the parameter values are known in this instance, and the parameter uncertainty is given by $\bar{\lambda}_1 = 0.00015$, $\sigma_{\lambda,1} = 0.0002$, $\bar{\lambda}_2 = 0.0003$, $\sigma_{\lambda,2} = 0.0001$, $\bar{\beta}_1 = 0.65$, $\sigma_{\beta,1} = 0.15$, $\bar{\beta}_2 = 0.6$ and $\sigma_{\beta,2} = 0.05$. (That is, we now have a hybrid of the instances from Sections 5.1.1–5.1.2 in which the uncertainty intervals for $\lambda$ match the uncertainty interval from Section 5.1.1, the uncertainty intervals for $\beta$ match the uncertainty intervals from Section 5.1.2, and all other parameters are common across all three instances.) The allocation of testing times across the subsystems for the optimistic solution is $(\tau_1^{\Phi=0}, \tau_2^{\Phi=0}) = (3860.4, 4899.6)$, for the robust solution is $(\tau_1^{\Phi=0.5}, \tau_2^{\Phi=0.5}) = (4656.7, 4103.3)$, and for the pessimistic solution is $(\tau_1^{\Phi=4}, \tau_2^{\Phi=4}) = (5656.9, 3103.1)$.

The pairwise hypervolume proportions for the optimistic and pessimistic solutions are respectively 82.48% and 65.78%, respectively. The robust solution’s reliability may now exceed the optimistic (pessimistic) solution’s by as much as 0.00981 (0.01317) and the potential magnitude by which the optimistic (pessimistic) solution’s reliability could exceed the robust solution’s reliability remains relatively small at 0.00581 (0.00735).

5.1.4 Series System with $N = 6$, Uncertainty in Both $\lambda$ and $\beta$

We now consider a series system with 6 subsystems. The parameters of this instance are represented in Table 1 with the exception that $M_\ell = 1$, $\ell = 1, \ldots, N$. We set $t = 4380$ and $b_1 = 8760$. The approximate percentile plot is given in Figure 4(a). For this instance, the pairwise hypervolume proportion for the optimistic and pessimistic solutions are 86.29% and 71.61%. The robust solution’s
reliability may, in this case, exceed that of the optimistic (pessimistic) solution by as much as 0.01163 (0.00656) and may be exceeded by as much as 0.00217 (0.00609). The testing times for each method are presented in Table 2. In this case, the testing times of the robust solution for subsystems 2, 4 and 6 fall between the pessimistic and optimistic solutions.

5.1.5 Series-Parallel System with $N = 6$, Uncertainty in Both $\lambda$ and $\beta$

We now perform the same set of analyses for a series-parallel system with 6 subsystems. The parameters are presented in Table 1, and we consider $b_1 = 8760$ and $t = 18,250$ for this instance. The pairwise hypervolume proportions for the optimistic and pessimistic deterministic approaches are 83.59% and 68.28%, respectively. We also present the percentile plot of the reliability difference between the robust solution and the optimistic and pessimistic solutions in Figure 4(b). In this case, the robust solution’s reliability may now exceed that of the optimistic (pessimistic) solution by as much as 0.02095 (0.01475) and may not be exceed by more than 0.00563 (0.01101). Thus, in moving from the series to the series-parallel instance, the robust optimization approach seems even more favorable as compared to deterministic methods. The solutions generated by each method

---

Table 1: Parameters for the series instance with 6 subsystems.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\lambda_\ell$</th>
<th>$\sigma^\ell_\lambda$</th>
<th>$\beta_\ell$</th>
<th>$\sigma^\ell_\beta$</th>
<th>$M_\ell$</th>
<th>$\ell$</th>
<th>$\lambda_\ell$</th>
<th>$\sigma^\ell_\lambda$</th>
<th>$\beta_\ell$</th>
<th>$\sigma^\ell_\beta$</th>
<th>$M_\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00008</td>
<td>0.00003</td>
<td>0.65</td>
<td>0.07</td>
<td>3</td>
<td>3</td>
<td>0.0004</td>
<td>0.00007</td>
<td>0.75</td>
<td>0.02</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.68</td>
<td>0.05</td>
<td>1</td>
<td>4</td>
<td>0.00025</td>
<td>0.00004</td>
<td>0.7</td>
<td>0.04</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0.00004</td>
<td>0.00003</td>
<td>0.65</td>
<td>0.1</td>
<td>2</td>
<td>6</td>
<td>0.00006</td>
<td>0.0002</td>
<td>0.6</td>
<td>0.15</td>
<td>1</td>
</tr>
</tbody>
</table>

$V = 1$, $c^\ell = 1$, $\tau^{\ell}_u = 1$, $\ell \in \{1, \ldots, 6\}$

---

Table 2: Testing times for the series instance with 6 subsystems.

<table>
<thead>
<tr>
<th></th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$\tau_4$</th>
<th>$\tau_5$</th>
<th>$\tau_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimistic ($\Phi = 0$)</td>
<td>692.1</td>
<td>3625.4</td>
<td>1554.3</td>
<td>2010.5</td>
<td>414.2</td>
<td>463.5</td>
</tr>
<tr>
<td>Robust ($\Phi = 0.5$)</td>
<td>658.3</td>
<td>3434.8</td>
<td>1476.8</td>
<td>1908.7</td>
<td>394.0</td>
<td>887.4</td>
</tr>
<tr>
<td>Pessimistic ($\Phi = 12$)</td>
<td>707.5</td>
<td>2707.5</td>
<td>1612.6</td>
<td>1633.0</td>
<td>544.3</td>
<td>1555.1</td>
</tr>
</tbody>
</table>
are presented in Table 3. The testing times for subsystems 2, 3 and 6 lie between the deterministic solutions.

5.2 Generating Uncertainty Intervals Based on Preliminary Test Data

We now demonstrate, using the three-subsystem instance defined in Table 4, how one may construct uncertainty intervals based on preliminary test data. In this set of results, we simulate the underlying failure/growth process (i.e., the nonhomogenous Poisson process described by Equation (2)) in order to generate initial failure data that may be used in planning future tests. Given this data, we compare the robust approach against deterministic approaches for generating test plans.

Initially, we simulate the AMSAA failure process (using the underlying $\lambda_\ell$- and $\beta_\ell$-values) for $\tau_\ell^0 = \tau_{\text{init}}$ hours for each $\ell = 1, \ldots, N$. (We consider the values $\tau_{\text{init}} \in \{6570, 7300\}$ in our experiments.) Based upon the results of the simulation, for each $\ell = 1, \ldots, N$, we derive maximum likelihood estimates (denoted $\hat{\beta}_\ell$ and $\hat{\lambda}_\ell$) of $\lambda_\ell$ and $\beta_\ell$ as

\[
\hat{\beta}_\ell = \frac{n_\ell}{n_\ell \ln(\tau_\ell^0) - \sum_{i=1}^{n_\ell} \ln(T_{\ell,i})}, \quad \text{(53a)}
\]

\[
\hat{\lambda}_\ell = \frac{n_\ell}{(\tau_\ell^0)^{\frac{n_\ell}{\hat{\beta}_\ell}}}, \quad \text{(53b)}
\]

Table 4: Parameters for three-subsystem series instance.

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\beta_1$</th>
<th>$\lambda_2$</th>
<th>$\beta_2$</th>
<th>$\lambda_3$</th>
<th>$\beta_3$</th>
<th>$t$</th>
<th>$V$</th>
<th>$c_1$</th>
<th>$b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.65</td>
<td>0.02</td>
<td>0.6</td>
<td>0.025</td>
<td>0.7</td>
<td>120</td>
<td>1</td>
<td>1</td>
<td>30,660</td>
</tr>
</tbody>
</table>
for each \( \ell = 1, \ldots, N \) and in which \( n_{\ell} \) is number of failures for subsystem \( \ell = 1, \ldots, N \) in the simulation, and \( T_{\ell,i} \), \( i = 1, \ldots, n_{\ell} \) is the \( i \)th failure time for subsystem \( \ell = 1, \ldots, N \). We then solve the deterministic allocation model (i.e., Model (1)) assuming \( \lambda = \hat{\lambda} \) and \( \beta = \hat{\beta} \) to allocate test times to the three subsystems. We refer to this deterministic approach as the max-likelihood approach. Using the test times \( \tau^\star_\ell \), \( \ell = 1, \ldots, N \), that result from the max-likelihood allocation model, we then evaluate \( R(\tau^\star; \lambda, \beta) \)—the true reliability after implementing the optimization model’s recommended test times—and report this value for comparison. Each subsystem’s failure times are generated based on the homogenous Poisson process by the failure rate given in Equation (2), and then used to estimate \( \hat{\lambda} \) and \( \hat{\beta} \). Because the estimates \( \hat{\lambda} \) and \( \hat{\beta} \) are random, we replicate this procedure 100 times (using the same underlying parameters given in Table 4).

To demonstrate the effect of our robust model, we repeat the above process using the robust model instead of the deterministic model to generate \( \tau^\star \). Specifically, given the results of the initial \( \tau^{\text{init}} \) hours of testing completed on each subsystem, we first construct \( 100(1-\alpha)\% \) two-sided confidence intervals (using the Fisher Matrix bound [32]) on each \( \lambda_\ell \) and \( \beta_\ell \), \( \ell = 1, \ldots, N \), as

\[
\lambda_\ell \in \left[ \hat{\lambda}_\ell e^{z_\alpha/2 \sqrt{\text{Var}(\hat{\lambda}_\ell)/\hat{\lambda}_\ell}}, \hat{\lambda}_\ell e^{-z_\alpha/2 \sqrt{\text{Var}(\hat{\lambda}_\ell)/\hat{\lambda}_\ell}} \right],
\]

\[
\beta_\ell \in \left[ \hat{\beta}_\ell e^{z_\alpha/2 \sqrt{\text{Var}(\hat{\beta}_\ell)/\hat{\beta}_\ell}}, \hat{\beta}_\ell e^{-z_\alpha/2 \sqrt{\text{Var}(\hat{\beta}_\ell)/\hat{\beta}_\ell}} \right],
\]

in which \( z_\alpha \) is the value of Normal distribution with lower tail probability \( \alpha \), and

\[
\text{Var}(\hat{\lambda}_\ell) = \frac{\hat{\lambda}_\ell^2 \left( n_\ell - \hat{\lambda}_\ell n_\ell \beta_\ell \ln(n_\ell)^2 \beta_\ell^2 \right)}{n_\ell^2 - \hat{\lambda}_\ell \beta_\ell^2 n_\ell \ln(n_\ell)^2 - (\hat{\lambda}_\ell \beta_\ell n_\ell \ln(n_\ell))^2},
\]

\[
\text{Var}(\hat{\beta}_\ell) = \frac{n_\ell \beta_\ell^2}{n_\ell^2 - \hat{\lambda}_\ell \beta_\ell^2 n_\ell \ln(n_\ell)^2 - (\hat{\lambda}_\ell \beta_\ell n_\ell \ln(n_\ell))^2}.
\]

We then define an instance of the robust optimization model (i.e., Model (11)) by letting the confidence intervals (55) play the role of Equation (7) in the construction of the uncertainty set \( U_\Phi \). We repeat this for all combinations of the values \( \alpha \in \{0.6, 0.8\} \) and \( \Phi \in \{0.3, 0.5\} \). For each of these combinations, we run 100 replications (reusing the same values of \( n_\ell \) and \( T_{\ell,i} \) generated in replicating the max-likelihood approach) because the uncertainty intervals are also random due to depending on the simulated failure data. Because \( 0 < \beta_\ell < 1 \) in the AMSAA model, we use 0.99 in place of \( \beta_\ell + \sigma_\ell^B \) in the robust optimization model if \( \beta_\ell + \sigma_\ell^B \geq 1 \). Likewise, we truncate the values of \( \beta_\ell \) and \( \hat{\beta}_\ell \) at 0.99 when they would otherwise appear in an optimization model with value at least 1.

In order to compare against what would be possible prior to this paper, we also compare against some deterministic approaches. The robust optimization model corresponding to \( \Phi = 2N = 6 \)
allows all parameters to take on their worst-case values (i.e., \( \lambda_\ell = \bar{\lambda}_\ell + \sigma^L_\ell \) and \( \beta_\ell = \bar{\beta}_\ell + \sigma^B_\ell \), \( \forall \ell = 1, \ldots, N \)) and thus corresponds to the pessimistic deterministic approach, which we refer to as the \textit{pessimistic approach}. The robust optimization model also becomes deterministic when \( \Phi = 0 \) as all parameters take their optimistic values (i.e., \( \lambda_\ell = \bar{\lambda}_\ell \) and \( \beta_\ell = \bar{\beta}_\ell \), \( \forall \ell = 1, \ldots, N \)). We refer to this as the \textit{optimistic approach}. We also compare against the deterministic model where all \( \gamma^L_\ell \) and \( \gamma^B_\ell \) are fixed equal to 0.5, and refer to the resulting approach as \textit{midpoint approach}. For each robust approach (i.e., for \( \Phi = 0.3 \) and \( \Phi = 0.5 \)) and each deterministic approach (i.e., max-likelihood, pessimistic, optimistic and midpoint), we record the true reliability that results from each of the 100 replications.

The results of our experiments are summarized in Table 5. In Table 5, \( \text{R}>\text{L} \), \( \text{R}>\text{O} \), \( \text{R}>\text{M} \), and \( \text{R}>\text{P} \) represent the number of times from 100 replications in which the robust approach respectively yielded higher reliability than the max-likelihood, pessimistic, optimistic, and midpoint approaches. Regardless of the value of \( \alpha \) and \( \tau_{\text{init}} \), both of the robust treatments are preferable to all of the deterministic treatments in more than half of the replications.

Figure 5 compares the reliability percentiles of the robust and deterministic approaches for \( \tau_{\text{init}} \in \{6570, 7300\} \) and \( \alpha \in \{0.6, 0.8\} \). Across 100 replications, we obtain 100 different system reliabilities for each method. We sort those reliabilities, and plot them in Figure 5, in which the smallest is in the far left and the largest is in the far right. Figure 5 suggests that the robust approaches are particularly effective in comparison to the deterministic approaches in the lower percentiles although the optimistic and max-likelihood approaches yield similar reliabilities as robust approaches for percentiles above 0.5.

Table 6 compares the average testing times (across the 100 replications) of the robust and the deterministic approaches for \( \tau_{\text{init}} \in \{6570, 7300\} \) and \( \alpha \in \{0.6, 0.8\} \). The testing times shown in Table 6 are the averages of 100 replications. One observation from Table 6 is that \( \tau_3 \) is larger for the robust approaches than the deterministic approaches. We conjecture this is related to the fact that \( \lambda_3 \) is larger than either \( \lambda_1 \) or \( \lambda_2 \), and \( \beta_3 \) is larger than either \( \beta_1 \) or \( \beta_2 \), so the third subsystem needs more testing time to reach a particular reliability. The robust solution hedges against potentially larger values of \( \lambda_3 \) and \( \beta_3 \) by allocating more testing time.
Table 6: Average testing time allocation for the series instance.

<table>
<thead>
<tr>
<th>$\tau_{\text{init}}$</th>
<th>$\alpha$</th>
<th>Max-likelihood</th>
<th>Optimistic</th>
<th>Midpoint</th>
<th>Pessimistic</th>
<th>$\Phi = 0.3$</th>
<th>$\Phi = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6570</td>
<td>0.6</td>
<td>7783.6</td>
<td>7296.0</td>
<td>8762.9</td>
<td>8643.2</td>
<td>7425.5</td>
<td>7476.3</td>
</tr>
<tr>
<td></td>
<td>$\tau_1$</td>
<td>7854.8</td>
<td>8064.5</td>
<td>8824.5</td>
<td>9314.0</td>
<td>7737.1</td>
<td>7824.1</td>
</tr>
<tr>
<td></td>
<td>$\tau_2$</td>
<td>8827.2</td>
<td>8505.5</td>
<td>9539.9</td>
<td>10093.8</td>
<td>8136.9</td>
<td>8147.1</td>
</tr>
<tr>
<td></td>
<td>$\tau_3$</td>
<td>13478.0</td>
<td>14089.9</td>
<td>12295.5</td>
<td>11252.3</td>
<td>14786.0</td>
<td>14688.8</td>
</tr>
<tr>
<td>7300</td>
<td>0.6</td>
<td>8354.8</td>
<td>8258.2</td>
<td>8402.2</td>
<td>8811.8</td>
<td>8045.7</td>
<td>8035.3</td>
</tr>
<tr>
<td></td>
<td>$\tau_1$</td>
<td>8354.8</td>
<td>8258.2</td>
<td>8402.2</td>
<td>8811.8</td>
<td>8045.7</td>
<td>8035.3</td>
</tr>
<tr>
<td></td>
<td>$\tau_2$</td>
<td>8827.2</td>
<td>8571.1</td>
<td>8886.9</td>
<td>9432.3</td>
<td>8197.7</td>
<td>8128.5</td>
</tr>
<tr>
<td></td>
<td>$\tau_3$</td>
<td>13478.0</td>
<td>13830.7</td>
<td>13370.9</td>
<td>12416.0</td>
<td>14416.5</td>
<td>14496.2</td>
</tr>
</tbody>
</table>
Table 7: Parameters for 11 instances that study the effect of system structure in resource allocation.

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\sigma_{1\lambda}^1$</th>
<th>$\beta_1$</th>
<th>$\sigma_{1\beta}^1$</th>
<th>$\lambda_2$</th>
<th>$\sigma_{2\lambda}^2$</th>
<th>$\beta_2$</th>
<th>$\sigma_{2\beta}^2$</th>
<th>$\Phi$</th>
<th>$t$</th>
<th>$V$</th>
<th>$c_1^2$</th>
<th>$b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00005</td>
<td>0.00001</td>
<td>0.7</td>
<td>0.1</td>
<td>0.00006</td>
<td>0.00002</td>
<td>0.8</td>
<td>0.05</td>
<td>2</td>
<td>8760</td>
<td>1</td>
<td>1</td>
<td>720</td>
</tr>
</tbody>
</table>

Figure 6: (a) Testing time allocation and (b) reliability for the first set of instances.

5.3 Effect of System Structure on Resource Allocation

In order to compare the effect of the system structure on testing time allocation, we consider two sets of instances. Each instance considers a system with two subsystems. The first set of instances consists of one component in the first subsystem and $s \in \{1, \ldots, 6\}$ components in the second subsystem, and the second set of instances consists of $s \in \{1, \ldots, 6\}$ components in the first subsystem and one component in the second subsystem. The parameters for the instances are provided in Table 7, and we use $P_{\text{sub}} = 20000$ and $\epsilon_{\text{sub}} = 1 \times 10^{-4}$ in the series-parallel algorithm.

The results are summarized in Figures 6–7. The testing time allocation for the first set of instances is presented in Figure 6(a), and for the second set of instances in Figure 7(a). The second subsystem has larger $\lambda$- and $\beta$-values, so when both subsystems have only one component, the second subsystem needs more testing time to reach a given reliability and its testing time is greater. By increasing redundancies for subsystem $\ell \in \{1, 2\}$, the testing time allocated to subsystem $\ell$ decreases. The system reliabilities for the first and second set of instances are presented in Figure 6(b) and 7(b). Because the second subsystem has larger $\lambda$- and $\beta$-values, installing a redundant component in the second subsystem lessens the effect of the less favorable parameter values and thereby improves the system reliability more than it would in the first subsystem; therefore, the reliability in the first set of instances is larger than in the second set of instances.

5.4 Computational Performance of the Cutting-Plane Algorithm

We now solve five instances to demonstrate the performance of the cutting-plane algorithm for a series system. For each instance, we choose $t = 2190$, $\Phi = 0.5$, $V = 1$, $c_1^2 = 1$, $b_1 = 8760$ and $\tau^0_{\ell} = 1$, $\ell \in \{1, 2\}$. Moreover, $\beta$-values are selected from values 0.6, 0.65, 0.7, 0.75, and 0.8, each of which has an equal probability of being selected, and we select $\lambda$-values uniformly from intervals
Table 8: Results for series instances.

<table>
<thead>
<tr>
<th>Inst.</th>
<th>N</th>
<th>Reliability</th>
<th>Solving time (seconds)</th>
<th>λ range</th>
<th>σ^B_ℓ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0.958445</td>
<td>2</td>
<td>[0.00001, 0.00005]</td>
<td>0.00001</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.973587</td>
<td>2</td>
<td>[0.000005, 0.000008]</td>
<td>0.00002</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>0.939991</td>
<td>4</td>
<td>[0.000001, 0.000004]</td>
<td>0.00001</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.967938</td>
<td>8</td>
<td>[0.0000004, 0.0000007]</td>
<td>0.000003</td>
</tr>
<tr>
<td>5</td>
<td>400</td>
<td>0.904244</td>
<td>65</td>
<td>[0.00000001, 0.00000005]</td>
<td>0.0000001</td>
</tr>
</tbody>
</table>

presented in Table 8. We select σ^B_ℓ from the values 0.05 and 0.1, again with an equal chance of selection, and the values of σ^L_ℓ are presented in Table 8. According to Table 8, the cutting-plane algorithm is able to find the optimal solution in no more than a few seconds even for large-size problems.

We also solve five instances to compare the efficiency of the cutting-plane algorithm for series-parallel systems. In the following, we discuss the instance generation procedure for series-parallel instances. Each instance includes 4, 10, 40, 100 or 400 subsystems. We choose β-values from values 0.6, 0.65, 0.7, 0.75, and 0.8, and M_ℓ-values from values 2, 3, and 4, all with equal probability of selection. Each of the N λ-values is selected uniformly from the range described in Table 9. The value of σ^L_ℓ, ℓ = 1, ..., N, is presented in Table 9, and the value of σ^B_ℓ, ℓ = 1, ..., N, is chosen from values 0.05 and 0.1, each with equal probability of selection. For each instance, we consider initial testing time τ^0_ℓ = 1, t = 8760, V = 1, c^L_ℓ = 1, b_1 = 720, and Φ = 0.5. The time limit is equal to 1200 seconds, and we select ε = 1 × 10^{-7} and ε_{sub} = 1 × 10^{-4}. The value of P_{sub} is equal to 20000 for Instances s ∈ {1, ..., 3} and 5000 for Instances s ∈ {4, 5}. As demonstrated in Table 9, the cutting-plane algorithm is consistently able to identify near-optimal solutions for these instances within 20 minutes.
<table>
<thead>
<tr>
<th>Inst.</th>
<th>N</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Optimality gap</th>
<th>( \lambda ) range</th>
<th>( \sigma_L^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0.984892</td>
<td>0.984898</td>
<td>5.43 \times 10^{-6}</td>
<td>[0.000002, 0.00006]</td>
<td>0.00001</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.947247</td>
<td>0.947261</td>
<td>1.43 \times 10^{-5}</td>
<td>[0.000002, 0.00005]</td>
<td>0.00001</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>0.982389</td>
<td>0.982433</td>
<td>4.42 \times 10^{-5}</td>
<td>[0.000007, 0.000009]</td>
<td>0.000003</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.956155</td>
<td>0.956415</td>
<td>2.61 \times 10^{-4}</td>
<td>[0.000005, 0.000009]</td>
<td>0.000002</td>
</tr>
<tr>
<td>5</td>
<td>400</td>
<td>0.925610</td>
<td>0.926971</td>
<td>1.36 \times 10^{-3}</td>
<td>[0.000001, 0.000005]</td>
<td>0.000001</td>
</tr>
</tbody>
</table>

6 Conclusion

In this paper, we consider the robust allocation of testing resources across the subsystems of series and series-parallel systems. We assume that the failures of each component occur according to the AMSAA model with uncertain parameters within a bounded uncertainty set. We propose exact algorithms for both series and series-parallel systems based on a cutting-plane approach, and prove that the algorithm for series systems is convergent. We show how the robust optimization model can be constructed by building confidence intervals from preliminary test data and demonstrate, using a numerical example, that the robust optimization approach is less likely than available deterministic approaches to generate solutions that result in a low reliability. Moreover, the cutting-plane algorithm for series systems can find the optimal solution in couple of seconds, and the proposed algorithm for series-parallel systems can provides a tight optimality gap for a set of test instances.

Future research may seek to extend our research to consider cold standby redundancy and/or dependent components. Another follow-on research direction is to consider resource allocation within a multi-stage reliability growth model, in which the failure information is incorporated at end of each testing stage to have a better estimation of the AMSAA model parameters. Additionally, a follow-on investigation may consider a situation in which the budget-restricted uncertainty set becomes narrower during the testing process, making it a dynamic/sequential test-planning problem. Furthermore, future research may develop heuristic methods for obtaining high-quality solutions quickly, which would be especially helpful given the computational effort required to solve the series-parallel problem to provable optimality.

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References


